Achieve, Inc.

RECOMMENDATIONS TO THE NATIONAL ASSESSMENT GOVERNING BOARD ON ALIGNING THE 12TH GRADE NAEP WITH COLLEGE, WORKPLACE, AND MILITARY EXPECTATIONS

MATHEMATICS

MARCH 2006

Aligning the Twelfth Grade NAEP
with College, Workplace, and Military Expectations

Recommendations to the National Assessment Governing Board from Achieve, Inc.

MATHEMATICS

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Executive Summary

The National Assessment Governing Board (NAGB) is seeking to redesign the 12th Grade NAEP to assess students' readiness for college, work, and the military. As part of this process, NAGB asked Achieve, Inc. to examine the 2005 12th Grade NAEP Mathematics Framework in relation to the mathematical and quantitative demands of the postsecondary world. In particular, NAGB asked Achieve how to define student preparedness in this context and what changes should be considered in NAEP objectives, balance, and achievement levels. To address these questions, Achieve assembled a small panel of experts drawn from K-12, postsecondary, research, and policy communities. This is the report of that panel.

Preparedness for credit-bearing college courses, high-growth jobs, and military careers have in common a broad range of mathematical knowledge and skills that open doors to diverse careers. This common core includes (a) the mathematics required to qualify without remediation for credit-bearing courses in postsecondary education; (b) quantitative tools associated with data, computers, and probability; (c) broad competence in mathematical reasoning; and (d) ability to integrate multiple content areas in diverse contexts.

To meet the proposed goal of assessing preparedness, the twelfth grade NAEP Mathematics Framework should:

- Be revised and expanded to incorporate objectives that place special emphasis on cross-cutting, purposeful problems such as those illustrated in the ADP workplace tasks.
- Add a new scale to measure cross-cutting performance while continuing four separate scales for number properties and operations, measurement and geometry, data analysis and probability, and algebra. Each such scale should include a full range of item complexity. These five scales should be combined with prescribed weights for reporting a unified mathematics proficiency.
- Include items that span content from the 8th grade NAEP framework through precalculus and statistics (but not calculus). Elementary and advanced items that lie outside the central focus of preparedness expectations should be sampled lightly, but sufficiently to produce reliable estimates of students' performance at these levels.
- Include objectives concerning the appropriate use of technology in the solution of mathematical and quantitative problems.
- Continue to use the same three levels of complexity that are part of the 2005 NAEP, with the same 25%-50%-25% distribution pattern.
- Ensure that at least 15% of testing time is devoted to extended constructed response items and that no more than 50% of testing time is devoted to multiple choice items.
- Describe achievement levels in terms of *prepared* for credit bearing college courses, *partially prepared* (requiring one remedial mathematics course), and *well prepared* (may be prepared for calculus or comparable courses).

In order to properly report students' preparedness for a wide variety of high quality postsecondary career options, the 12th grade NAEP mathematics report should unify its separate scales into a single set of mathematics achievement levels that signify *preparedness on all five scales for all three postsecondary options*. Additionally, to aid education and policy leaders, NAEP should prepare achievement level descriptions and associated cut points for each of the three types of career trajectories on each of the five proposed NAEP scales.

Introduction

In March 2004, the National Assessment Governing Board (NAGB) received a report from a National Commission it had established to "review the current purpose, strengths, and weaknesses" of the 12th grade National Assessment of Educational Progress (NAEP).[1] In the fall of 2004, NAGB engaged Achieve, Inc. to respond to a major Commission recommendation that the 12th grade NAEP be redesigned to report on students' readiness for college-credit coursework, workforce training, and entrance into the military. Rather than merely asking how well students learned what they were taught in elementary and secondary school—NAEP's original purpose—the Commission urged that the 12th grade NAEP be redesigned to measure how well students are prepared for postsecondary pursuits.

To explore the implications of redesigning NAEP in this way, NAGB asked Achieve to prepare a concept paper that examines the 2005 NAEP Mathematics Framework at the 12th Grade level in relation to the mathematical and quantitative demands of college, military, and the world of work. This request followed a similar study that Achieve conducted for NAGB concerning the 12th grade NAEP Reading Framework. Following the lead of this prior study, Achieve sought to determine, for mathematics:

1. How NAEP should define student preparedness for college, training for employment, and entrance into the military.
2. What changes should be considered for the 2005 NAEP Framework to reflect the expectations of the American Diploma Project.
3. What changes NAEP should consider for item construction, content balance, and cognitive targets to better reflect postsecondary college and career demands.
4. What changes should be considered to NAEP's achievement level descriptions to enable meaningful reporting of student preparedness.

To address these questions, Achieve assembled a small panel of experts drawn from K-12, postsecondary, research, and policy communities. This is the report of that panel.

Background

Data from many sources suggest a major disjuncture between student aspirations and accomplishments. Although a high school diploma is a prerequisite to virtually all careers, only about two out of three Americans graduate from high school on time[2]—one of the lowest rates among industrialized nations.[3] On-time graduation rates of minorities are much worse: estimates suggest that only half of Hispanic and African-American students finish high school four years after entering ninth grade.[4] Less education translates directly into less earning power and a lower standard of living. In 2002 the median income of students without a high school diploma was

1 National Commission on NAEP 12th Grade Assessment and Reporting. *12th Grade Student Achievement in America: A New Vision for NAEP*. A Report to the National Assessment Governing Board, March 5, 2004.

2 Barton, Paul E. *One-Third of a Nation: Rising Dropout Rates and Declining Opportunities*. Princeton, NJ: Educational Testing Service, February, 2005. URL: www.ets.org/research/pic.

3 Organization for Economic Co-operation and Development. *Education at a Glance 2004*. Table A2.1: Upper Secondary Graduation Rates (2002). URL: www.oecd.org.

4 Jay P. Greene, *High School Graduation Rates in the United States*, New York: The Manhattan Institute, April 2002.

only $21,600, just 70% that of students who graduate from high school but who have no further education[5] and barely above the federal poverty guidelines for a family of four.

Paradoxically, at the same time as politicians and the public have espoused a goal of college for all, the graduation rate from high school has been slipping consistently since reaching a peak of 77% in 1969. This downward trend is not well known to parents or taxpayers, perhaps because the United Sates has no widely accepted and reliable way to monitor high school graduation rates. Nonetheless, a recent study by The Urban Institute concludes that "a growing body of work is pointing to a [four-year] graduation rate as low as 65 percent nationwide. In very large districts, in those educating large numbers of disadvantaged minority students, and in states with historically struggling educational systems, the odds of graduating from high school for the *average* student lie well below 50:50."[6] Since NAEP samples students who are in school, these data suggest that a large fraction of America's youth, disproportionately poor and minority, are omitted from the sampling frame of the12th grade NAEP.

What of those who do finish high school on time? Common sense suggests that a high school diploma should warrant adequate preparation for the demands of adult life. In reality, it falls far short of this goal. Employers estimate that 40% of high school graduates are inadequately prepared in mathematics while college faculty estimate that as many as half of all entering college students are not adequately prepared to succeed in college-level mathematics courses.[7] These views are shared by recent high school graduates themselves, 40-45% of whom report significant gaps in their preparation. Researchers from Stanford University's Bridge Project illuminate these data by documenting major discrepancies among high school expectations, state tests, college entrance tests and placement requirements.[8] This "fractured" system undermines student aspirations by sending mixed signals about what constitutes adequate preparation for postsecondary endeavors.

The benefit of education on earnings is enhanced significantly by a college degree: students with a bachelor's degree (but no higher degree) earn 60% more than students with only a high school diploma, and well over twice what high school dropouts earn. Yet four out of every ten high school graduates are unable to enroll in credit-bearing college courses without first passing one or more remedial courses.[9] Students who begin in remedial courses take longer to complete their degrees and have a much lower probability of doing so.[10] Moreover, remediation increases educational costs for parents, students, and taxpayers. Employers too pay a stiff price for the lack of academic preparation among workers. U.S. businesses expend billions of dollars each year for remedial education of their employees. A conservative estimate puts the total national cost of remediation at over $16 billion, two thirds of which is spent by business, one-third by educational institutions.[11]

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5 *Education Pays*. New York: The College Board, 2004, p. 10.

6 Christopher B. Swanson and Duncan Chaplin. "Counting High School Graduates when Graduates Count: Measuring Graduation Rates under the High Stakes of NCLB." Education Policy Center, The Urban Institute, February 25, 2003.

7 Achieve, Inc. *Rising to the Challenge: Are High School Graduates Prepared for College and Work*? February 2005.

8 Andrea Venezia, Michael W. Kirst, and Anthony L. Antonio. *Betraying the College Dream: How Disconnected K-12 Postsecondary Education Systems Undermine Student Aspirations.* Final Report of the Bridge Project, Palo Alto, CA: Stanford Institute for Higher Education Research, p. 3.

9 National Center for Education Statistics. *The Condition of Education 2001*, 2001.

10 Michael W. Kirst. "College Preparation and Grade 12 NAEP." Paper prepared for NAGB, September 2003, p. 4.

11 Jay P. Greene. *The Cost of Remedial Education*. The Mackinac Center. September 2000.

Mathematics has a magnifying effect on educational achievements; accordingly, it is central to the educational aspirations of students and of the nation. Studies regularly show a strong association between completing high school courses such as Geometry and Algebra II and future success in college and work.[12] Economists and politicians regularly call attention to the importance of strong science and mathematics as an engine of economic growth.[13] Notwithstanding these policy reasons for strengthening student preparation in mathematics, the real beneficiaries of strong mathematical preparation are students themselves.

Mathematics opens doors to careers, and in today's digital world it opens more doors than ever before. Mathematical methods that were once anchored in physics and engineering have spread to distant fields such as architecture, cinema, epidemiology, finance, genetics, and investment. The unprecedented mathematization of careers across the full spectrum of human endeavor has expanded the repertoire of important mathematics well beyond traditional school subjects of algebra, geometry and trigonometry. Tools from such areas as probability, statistics, graph theory, Boolean algebra, theory of algorithms, number systems, and matrix algebra—oftentimes implemented and visually portrayed using common computer tools—are as likely as advanced parts of algebra and geometry to be of value for high school graduates as they move on to employment and further study.

Redesigning the 12th grade NAEP from a retrospective assessment to a prospective measure of student preparedness would reveal the degree to which K-12 schools are fulfilling their goal of preparing students for postsecondary pursuits. A revised NAEP would provide clearer signals to educators and students about preparation necessary for life and work in contemporary society, as well as important data to policymakers on college and work preparedness. Existing instruments such as state high school graduation tests, SAT and ACT college admission exams, *WorkKeys*,[14] college placement tests, and AP (Advanced Placement) exams all serve different and more limited purposes. The 12th grade NAEP is ideally situated to fill the assessment gap between high school, college, and careers: it offers a snapshot of students' performance and understanding at the point of transition to adulthood and citizen responsibilities.

Parameters, Limitations, Caveats

Preparedness. The National Commission whose recommendation led to this study speaks of redesigning NAEP to report on readiness for college coursework, training for employment, and military service. Because the term "readiness" has a specific meaning with regard to reading in the elementary grades, the Achieve Reading Panel decided to use the term "preparedness" instead of "readiness" throughout their report. For consistency, this report from the Mathematics Panel follows the same convention.

Scope. The anchor of the American Diploma Project is a series of workplace tasks and postsecondary assignments that illustrate contexts in which high school graduates will be called on to demonstrate various kinds of mathematical proficiency. Employers and college educators alike stress the need for broad skills that are well integrated. Mathematical, quantitative, and logical

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12 The Education Trust. "A New Core Curriculum for All." *Thinking K-16.* 7:1 (Winer 2003) p. 15-16.
13 Lamar Alexander. "Nurturing the Next Einsteins." *Science* (18 February 2005) 1013.
14 ACT, Inc. *WorkKeys: Improving the Quality of America's Workforce*. URL: www.act.org/workkeys/.

challenges are typically embedded in contexts requiring many other skills such as communication (listening, reading, writing, speaking), interpersonal (leadership, teamwork), research (finding and verifying relevant information), and technological (using computer tools; sharing data electronically). These broad capabilities are not merely "add-ons," skills high school graduates need in addition to mathematical proficiency; they describe the way mathematics is actually done.

Ideally, a revised NAEP would assess students' abilities to use mathematics as it actually arises in college and work settings. But NAEP is not designed for this kind of task. Perforce, NAEP can assess only a limited range of problem solving skills. Thus this study focuses on potential ways of improving the nature and variety of problems that NAEP can assess, not on the penumbra of associated skills that are required for successful use of mathematics in work or college settings. Even a much transformed NAEP can monitor only a few of the multiple dimensions of mathematical proficiency that are important for postsecondary preparedness.

Reading. 12th grade NAEP involves two separate assessments—one in Mathematics (the subject of this study) and the other in Reading (the subject of a prior Achieve study). As just noted, external clients of mathematics stress the importance of broad connections that in most cases involve both mathematical and reading skills. But internally too, it is important to recognize and assess the deep connections between reading and mathematics. Three of the four components of reading stressed by that panel—language, logic, and informational text—often involve quantitative concepts and mathematical ideas. It is important to include appropriate quantitative and logical components in these aspects of reading assessments. Reciprocally, reading is an essential aspect of mathematics, not only in decoding "word problems" but also in understanding the meaning of mathematical sentences and in following a logical argument. The traditional separation of Mathematics from Reading in the NAEP assessment limits the opportunities for developing this connection as thoroughly as would be desirable.

Psychometrics. The design and analysis of a NAEP assessment requires extensive application of highly technical psychometric principles that are usually understood only by specialists. In some cases, application of these principles conflicts with good mathematics. Whereas mathematical practice most often requires integrative skills that cut across different topics and domains, psychometric principles favor "clean" items that are narrowly targeted on a single skill. Moreover, when a NAEP assessment is piloted, items whose responses do not satisfy psychometric assumptions (e.g., those that are not consistent with the assumed item response model) are often discarded. Yet these are very often the kinds of items that best reveal mathematical thinking: when confronted with conceptual questions requiring fresh thought, some "average" students may do better than others who rely on memory to get high marks. Panelists strongly agreed that whenever issues of this sort arise where good mathematics conflicts with psychometric tradition, good mathematical practice should prevail.

Backwash. Assessments do more than monitor performance: they also send signals that influence curriculum. As NAEP increases in importance as a calibrator of state efforts to improve education, districts and teachers will understandably seek to ensure that their programs match the NAEP framework. Since the scope of mathematics broadens significantly as students advance in grades, the match between the NAEP framework and school practice weakens significantly from 4th to 8th to 12th grade. In particular, high school mathematics is far more diverse in potential topics and approaches than is elementary or middle school mathematics. So the danger of restricting creative teachers and innovative programs is much greater with a powerful12th grade NAEP that may become high stakes for states. For this reason, it is especially important that the mathematical objectives of a revised 12th grade NAEP be sufficiently broad to encompass a full range of options appropriate to preparing students for college, work, and the military.

The American Diploma Project

NAGB asked Achieve to prepare this analysis in part because of the latter's recent efforts, in conjunction with The Education Trust and the Thomas B. Fordham Foundation, to establish a benchmark for the transition between secondary and postsecondary worlds. After a two-year study, the American Diploma Project (ADP) published a report that described performance expectations in English and mathematics that high school graduates must meet in order to succeed in credit-bearing college classes or in high growth jobs with strong career trajectories.[15] These descriptions include typical workplace tasks, postsecondary course assignments, and associated high school outcome expectations. One of ADP's key recommendations reflects the motivation for this very analysis: to revise the 12th grade NAEP so that it will provide information on the extent to which high school seniors are ready for college and work.

The ADP examples and expectations were developed with input from (a) leading economists who analyzed projections for jobs that pay enough to support a family and provide potential for career advancement; (b) frontline managers from 22 high-growth occupations about the skills they believed were most useful for their employees; and (c) postsecondary and business leaders from five partner states (Indiana, Kentucky, Massachusetts, Nevada, and Texas) concerning the knowledge and skills required for success in both college and work. These three strands of research converged on an ambitious set of outcome expectations in English and mathematics for all high school graduates—expectations that emphasize the strong analytical skills that high school graduates will need in order to be prepared for credit-bearing college courses and jobs with strong career trajectories. The ADP consensus among business and education leaders has been reaffirmed by a subsequent study of recent high school graduates who report by margins greater than 2:1 that knowing what they do now about the demands of work or college, they would have elected higher level and more challenging courses.[16]

Although the ADP benchmarks are divided into disciplines (English and mathematics), its report *Ready or Not* is replete with testimonials from employers and faculty stressing the futility of compartmentalized knowledge. Pleas for integrated performance are strongly supported by the workplace tasks assembled in the report, less so by the postsecondary assignments. The ADP study reveals that requisite skills are more tightly integrated at work than in typical postsecondary classroom assignments. Since real work generally requires the application of knowledge and skills from more than one content area, distinctions among subjects, according to ADP, "are not relevant to the workplace." Thus *Ready or Not* urges educators to "anchor academic standards in the real world."

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[15] The American Diploma Project. *Ready or Not: Creating a High School Diploma that Counts*. Washington, DC: Achieve, Inc., 2004. (URL: http://www.achieve.org/achieve.nsf/AmericanDiplomaProject)

[16] Achieve, Inc. *Rising to the Challenge: Are High School Graduates Prepared for College and Work*? February 2005.

Mastery of individual skills without understanding their connections across content areas is, according to ADP, "inconsistent with what is expected beyond high school." To be successful a high school graduate must be able to (a) blend knowledge and skills from many areas to identify, formulate and solve problems; (b) connect new information to existing knowledge; and (c) access and assess knowledge from a variety of sources delivered through a variety of media. Further, workplace tasks tend to involve longer-term collaborative projects in which an individual contributes to a group effort. "Cooperation demands greater versatility in communication—in listening and speaking, in reading and writing—than typically is required in writing a paper or solving a problem as part of a traditional course assignment."

Fifteen years ago the influential SCANS report *What Work Requires of Schools* made a similar argument, stressing the importance for work of five broad competencies (productive use of resources, information, systems, technology, and interpersonal skills) built on a foundation of basic skills, thinking skills, and personal qualities.[17] This framework slices across the disciplinary organization of schools and tests, intersecting mathematical habits of mind in novel but vital contexts (e.g., allocation of time and space; acquiring and evaluating data; designing and improving systems; making decisions and solving problems). Since then, the global spread of technologically based information systems has significantly increased the importance of these cross-cutting mathematical dispositions and habits of mind.

To capture the complex nature of mathematical preparation needed for contemporary postsecondary endeavors, ADP offers two perspectives: tasks and assignments that illustrate the embedded character of mathematical practice, and a framework of specific expectations required to perform these tasks and assignments. ADP's mathematical expectations (reproduced in Appendix A) are organized under four strands that closely parallel the areas that NAEP uses to arrange its own objectives:

- *Number Sense and Numerical Operations*: Compute fluently and accurately without a calculator; number systems (integers, rational, real, and complex numbers); calculators and computers; estimation; spreadsheets.
- *Algebra*: Perform basic algebraic operations fluently and accurately; use and graph functions, equations, and inequalities; relate algebraic properties of functions to geometric properties of their graphs; solve verbal problems; analyze mathematical models.
- *Geometry*: Understand axioms, definitions and theorems; prove theorems; straight edge and compass constructions; congruence and similarity; properties of a circle and of special right triangles; Pythagorean theorem and its converse; rigid motions (reflections, translations and rotations); geometric measurements and designs; scale factors; solids and surfaces in three-dimensions; coordinate geometry; and right-triangle trigonometry.
- *Data Interpretation, Statistics & Probability*: Explain and apply quantitative information; critique alternative ways of presenting and using information; use data and statistical thinking to draw inferences, make predictions and justify conclusions.

In addition, ADP calls attention to several aspects of mathematical reasoning that are to be "woven throughout" the four content areas, for example, using inductive and deductive reasoning;

[17] Secretary's Commission on Achieving Necessary Skills. *What Work Requires of Schools: A SCANS* Report for America 2000. Washington, DC: U.S. Department of Labor, 1991.

understanding the role of definitions, proofs, and counterexamples; distinguishing relevant from irrelevant information; using mathematical modeling; devising independent checks of accuracy; and using examples effectively. These ADP reasoning expectations are listed in Appendix B.

The goal of these objectives is primarily to enumerate what is required to prepare students for credit-bearing courses in college (and in career-trajectory jobs). In addition, the ADP report *Ready or Not* also includes some content objectives (marked with asterisks) that are required for those students who intend to take calculus as their first college mathematics course.

The 12th Grade NAEP Mathematics Framework

In 1990 and 1992, NAEP mathematics tests were organized by means of a two-dimensional "content by mathematical ability" matrix design. Subsequently, in 1996 and 2000, the NAEP mathematics framework was expanded to a three dimensional design based on "content areas by mathematical abilities by mathematical power." (The dimension called "mathematical abilities" referred to conceptual understanding, procedural knowledge, and problem solving, while "mathematical power" referred to reasoning, connections, and communication.) Revisions incorporated in the latest (2005) framework sought to reflect recent curricular emphases and to provide more specific objectives at each grade level.[18]

The 2005 NAEP Framework continues the same content areas used by NAEP in 1996 and 2000 (and by most states in their own mathematics standards and assessments). These NAEP areas and corresponding percentage weights are listed in the following table, together with the corresponding ADP strands for comparison:

NAEP Weight	2005 NAEP Framework	ADP Benchmarks
10%	Number Properties and Operations	Number Sense and Numerical Operations
30%	Measurement and Geometry	Geometry
25%	Data Analysis and Probability	Data Interpretation, Statistics & Probability
35%	Algebra	Algebra

While detailed objectives within the strands often differ both in language and in purpose, the overall mathematics frameworks of ADP and NAEP are remarkably similar. (Appendix C provides a complete list of NAEP's 2005 12th grade mathematics objectives.)

NAEP's relatively light emphasis on number and operations is based on the assumption that "the majority of work [in this strand] is done in the context of the other content areas." Measurement is combined with geometry since "the majority of measurement topics suitable for 12th-grade students are geometrical in nature." (Most panelists disagree with this assertion.) These categories are "not intended to suggest that mathematics itself can be separated into discrete elements," but to ensure that "important mathematical concepts and skills are assessed in a balanced way." Each NAEP item assesses an objective that can be associated with a primary content area of mathematics.

[18] National Assessment Governing Board. *Mathematics Framework for the 2005 National Assessment of Educational Progress.* U.S. Department of Education, September 2004.

Each item also makes certain demands on students' thinking; these cognitive demands form the second dimension of the 2005 NAEP Mathematics Framework. The 2005 Framework compresses the two dimensions formerly called "mathematical abilities" and "power" into the single dimension called "mathematical complexity." This new dimension is based on the properties of an item rather than on the abilities of a student; it answers the question, "What does the item ask of the student?" (for example, recall a basic fact, apply a skill in a common situation or in an innovative context, develop a non-routine solution, or derive an original proof). The levels in this dimension are ordered from low to high, simple to sophisticated, routine to novel. The categories of complexity (low, moderate, high) form an ordered description of the demands an item may make on a student, ranging from recall (low) to making connections (moderate) to analyzing assumptions (high). (See Appendix D.) Ordering of the complexity levels is "not intended to imply a developmental [or pedagogical] sequence," but a "description of the different demands made on students" by particular test items.

Subject to constraints imposed by the timed nature of the NAEP assessment, the 2005 NAEP Framework suggests that half of the total possible score be based on items of moderate complexity, with the remainder based equally on items of low and high complexity. Similarly, the Framework recommends that approximately half of a student's testing time be allotted to multiple-choice items, with the remaining half devoted to constructed response items, both short and extended.

By tradition and policy, two-thirds of the NAEP mathematics tests at each grade level assess students' mathematical knowledge and skills without access to a calculator, while one-third of each test allows the use of a calculator. With few exceptions, 12th grade students can use whatever calculator—graphing or otherwise—they are accustomed to. Questions on the calculator portion of the test are designed with calculator use in mind; however, the 2005 Framework specifies that no items on the 2005 12th grade assessment should provide an advantage to students with a *graphing* calculator. (Panelists note that because of their widespread use in secondary schools, familiarity with graphing calculators is now a common expectation for students in beginning college mathematics courses. Moreover, NAEP items could assess the degree of such familiarity without actually expecting that students use a graphing calculator during the test.)

NAEP uses item response theory (IRT) to create a mathematics scale that portrays what students *actually* know and can do. (IRT is a statistical method that orders assessment items by difficulty on a single scale that corresponds, theoretically, to some latent trait or underlying construct.) To report what students *should* know and be able to do, NAEP uses achievement levels that are developed by a broadly representative panel of teachers, education specialists, and members of the general public.

For many years, NAEP has reported results in terms of three achievement levels: basic, proficient, and advanced. *Basic* denotes "partial mastery of prerequisite knowledge and skills" that are fundamental for proficient work. *Proficient* represents solid academic performance that demonstrates competency over challenging subject matter, "including subject-matter knowledge, application of such knowledge to real-world situations, and analytical skills appropriate to the subject matter." *Advanced* represents "superior performance." Full descriptions of the current levels for mathematics are contained in Appendix E. (Panelists note that the description of "advanced" achievement in Appendix E is not at all advanced in relation to performance expectations (and accomplishments) of many twelfth grade students. Nor does it reflect anything like an advanced level in terms of international performance.)

Parts of the preparedness domain that NAEP seeks to document have been explored for many years through ACT's WorkKeys program.[19] WorkKeys is an assessment system designed to measure skills needed for today's jobs with the aim of identifying gaps which can then be filled by targeted educational programs. WorkKeys targets a variety of job skills including reading, writing, listening, and teamwork. One test covers applied mathematics and is structured in five difficulty levels numbered 3 through 7. Items at Level 3 are those that translate easily from a word problem to an equation; all necessary information, and no more, is presented in logical order. At level 7 content or format may be unusual, information may be incomplete or implicit, and problems generally involve multiple steps of logic and calculation.

In 2003 ACT collaborated with NAGB to study the relation of WorkKeys to 12th grade NAEP.[20] This study concluded that in mathematics there is "100% agreement" between WorkKeys difficulty levels and NAEP achievement levels. WorkKeys levels 3-5 all correspond to NAEP's Basic; WorkKeys level 6 is mostly NAEP Proficient with some Advanced, and level 7 is mostly Advanced with some Proficient. The major difference is in balance: skills at WorkKeys levels 3 and 4 that are important to the workplace are not assessed by the current 12th grade NAEP, and neither are items situated in "workplace-focused contexts." For these reasons this study concluded that the current 12th grade NAEP could not be used to make any general statements about students' readiness for work.

Question 1. How should NAEP define student preparedness?

As NAEP considers the shift from reporting on students' achievement in relation to a relatively stable curriculum to reporting on students' preparedness for postsecondary pursuits in a world of rapidly changing expectations, this first question takes on paramount importance. Answers to all subsequent questions addressed in this paper depend on the answer given to this one. However, as the Mathematics Panel began to examine this question, several difficult issues emerged:

Issue 1: *Is mathematical preparedness the same for work as it is for college?*

The panel spent some time discussing what it means for a high school graduate to be "prepared" for college, work, and the military. Central to this discussion is the question of whether it makes sense to talk about preparedness as a single goal for all students in all circumstances. Historically, academic and vocational programs have followed different tracks that diverged significantly. In mathematics, the academic track leads to calculus, progressing en route through algebra, geometry, trigonometry, and precalculus. Vocational programs typically emphasize practical applications of arithmetic; many students in these programs never study algebra or geometry in

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[19] ACT, Inc. *WorkKeys: Improving the Quality of America's Workforce*. URL: www.act.org/workkeys/.

[20] "Analysis of Commonalities and Gaps: Comparing the National Assessment of Educational Progress and ACT WorkKeys Assessments." NAGB Working Paper, September 2003.

any systematic manner. (Likewise, many students in the academic track never learn any practical applications of the mathematics they study.)

But today requirements of work are much different. The ubiquitous role of computers, the demands of high performance industries, the flattening of managerial structures, and the emphasis on quality control all require employees with greater mathematical skills than ever before. As our nation shifts gears from a labor-intensive manufacturing economy to an internationally competitive knowledge economy, mathematics has become, as Robert Moses' famously put it, "the new civil right."[21] The ADP study provides substantial evidence of how the workplace is changing in ways that require increasing levels of mathematical and quantitative skills.

Although there are noticeable differences between the ideal preparation for college and work—formal proofs are more important for college, quality control charts for work—the overlap in required mathematics is enormous, and inseparable. Like colleges, employers devote enormous resources to remedial education just to prepare employees for the regular on-the-job training that is essential in today's fast-paced economy. Virtually all career-trajectory jobs (and comparable military assignments) require some type of training beyond high school. All students need to be prepared for such training. Although the mathematics used on the job may differ from that used in the classroom, the mathematics needed to avoid remediation is essentially the same in all cases.

At the same time as the academic demands of career preparation have increased, so has the importance of practical quantitative reasoning. Down to earth skills such as estimating risks, drawing inferences from data, managing budgets, understanding variability, and distinguishing correlation from causation are of increasing importance for all students. These ubiquitous kinds of mathematical and quantitative reasoning are part of the large overlap in mathematical preparedness for college and work. Similarly, preparation for credit-bearing college courses in mathematics includes most of the competencies needed for successful performance in high growth jobs and the military, although such high growth jobs probably require more experience in integrating problem solving across traditional course boundaries than do college mathematics courses.

***Resolution*:** Preparedness for credit-bearing college courses, high-growth jobs, and military careers have in common a broad range of mathematical knowledge and skills that open doors to diverse careers. This common core requires content from traditional high school algebra and geometry, newer tools associated with data, computers, and probability, broad competence in mathematical reasoning, and experience integrating multiple content areas as exemplified by the ADP workplace tasks.

Issue 2: *What is the standard for mathematical preparation for credit-bearing college courses?*

At first glance, mathematics appears to be the discipline least amenable to policies based on a single standard of preparedness. Both in scope and in depth—what NAEP calls content and complexity—students' mathematical preparation for postsecondary endeavors is extraordinarily diverse. Most universities place entering students in six or seven different levels of sequenced

[21] Robert P. Moses (with Charles E. Cobb, Jr.). *Radical Equations: Civil Rights from Mississippi to the Algebra Project*. Boston, MA: Beacon Press, 2001.

courses ranging from extreme remediation ("arithmetic for college students") to very advanced placement (abstract algebra and real analysis). Between these extremes lie 5-6 years of study. Large undergraduate enrollments can be found at each level. Approximately one-third of enrollments in undergraduate mathematics are remedial, that is, at or below the level of what colleges call intermediate algebra and what high schools usually call Algebra II. Another third comprise college algebra, trigonometry, and precalculus—the typical 4th year of high school mathematics taught in college to students who either never studied, never learned, or have forgotten its content. Only one-third of higher education mathematics enrollments are in courses at or above the level of calculus.[22] Studies of remediation in higher education have concluded that because of the enormous variety of postsecondary institutions, the United States has "no consistent standards about what constitutes college-level work;" as mathematics enrollment data show, remediation is clearly "a core function" of higher education.[23]

In setting admission policies, different colleges define mathematical preparedness in different ways. Most require two or three years of high school mathematics for general admission; some require a fourth year for admission to special programs (e.g., engineering) and a few selective universities have established a four-year mathematics requirements for all new students. Due to the increasing variability of high school mathematics courses, many institutions have created more explicit statements about mathematical preparation; examples include the University of California and the California State University System,[24] the University of Minnesota (on behalf of all postsecondary institutions in Minnesota),[25] and the *Standards for Success* project involving more than 25 research universities.[26] Typically, these statements include an advanced layer recommended for students who intend to pursue quantitatively rich subjects in college.

ADP, too, uses a system of asterisks to identify a selection of more advanced topics that are required for students who "plan to take calculus in college." The presence of the asterisked items in ADP reflects agreement with the position of higher education that proper preparation for calculus—itself a prerequisite for many majors—requires approximately a year of mathematical study beyond what is generally necessary to be prepared for credit-bearing college mathematics courses. (It should be noted, however, that the asterisked items in ADP are uneven in scope and inadequate as preparation for calculus. In mathematical jargon, they are necessary but not sufficient for this goal.)

This discussion is of much more than just academic interest. Whole industries are devoted to the task of identifying and treating differences in student capabilities at the transition from secondary to postsecondary education. Indeed, the United States prides itself on an incredibly diverse and robust system of postsecondary education, with nearly 10,000 public and private institutions ranging from specialized institutes offering technical certificates to world-class research

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22 David Lutzer, et al. *CBMS 2000: Statistical Abstract of Undergraduate Programs in the Mathematical Sciences in the United States*. Providence, RI: American Mathematical Society, 2002, p. 11.

23 Ronald Phipps. *College Remediation: What it Is; What it Costs; What's at Stake*. Washington, DC: The Institute for Higher Education Policy, 1998.

24 "Statement on Competencies in Mathematics Expected of Entering College Students." Oakland, CA: University of California, June 1997. (URL: www.universityofcalifornia.edu/senate/reports/mathcomp.html)

25 "College Mathematics Competence." Minneapolis, MN: University of Minnesota, 2004. (URL: http://evpp.jaws.umn.edu/math/)

26 David T. Conley. *Understanding University Success*. Eugene, OR: Center for Educational Policy Research, University of Oregon, 2003.

universities.[27] Admission to most of these programs requires no more than a high school diploma, and in some cases not even that. Indeed, despite wide differences in secondary school performance, most postsecondary certificate and diploma programs do succeed in launching many students onto productive careers. Calculus is far from the only gateway to successful careers, although virtually every route requires some type of quantitative skill.

***Resolution*: ** To be fully prepared for credit-bearing college courses—both in mathematics and in other fields—students need to enter college with operational competence in a broad variety of mathematics including numbers, measurement, ratios, graphs, data, estimation, triangles, circles, sets, variables, functions, equations, inequalities, probability, coordinate systems, and logical reasoning.

Students who enter postsecondary education able to employ and integrate this knowledge and these skills will be prepared for the lowest level credit-bearing college courses both in mathematics (e.g., college algebra or introductory statistics) and in other subjects such as economics and chemistry that depend strongly on mathematical methods. This level of preparation matches closely the mathematics graduation expectation in states with the strongest current requirements. In the future such an expectation may also become the standard for high stakes state tests, although most states are very far from this goal at the present time.[28]

It is important to recognize that the mathematical prerequisite for admission to and success in credit-bearing college courses is approximately one year short of what students need to be prepared for calculus. Notwithstanding the national need for scientists and engineers, it is unrealistic now and for the foreseeable future to expect that *all* high school graduates would be prepared to study calculus as their first college mathematics course. (Only engineering colleges and institutes of technology currently expect such preparation, although increasing numbers of colleges and universities have decided to raise their mathematics admission requirement from three to four years of high school mathematics.) Moreover, of course, calculus is not necessarily the best college mathematics course for all students. Students who are well prepared for college algebra and introductory statistics will not need remediation in college and will be well prepared for most fields that do not depend on calculus.

That said, to maintain fluency it is very important for college-bound students to study mathematics in the senior year of high school. Paradoxically, but commonly, students who meet the mathematics preparedness standard as sophomores or juniors and who then take no more mathematics in high school often find themselves mathematically unprepared when they enter college and find themselves required to take remedial courses covering topics that they once learned but have since forgotten.

Finally, the panel took note of contradictions inherent in the transition from secondary to postsecondary education that often work against student success. College and university admission standards are usually stated in very broad language, so many students who nominally meet these standards have substandard preparation in mathematics. Sometimes the stated

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[27] Patricia Q. Brown. *Postsecondary Institutions in the United States: 1993-94 and 1998-99*. NCES 2001-176. Washington, DC: U.S. Department of Education. National Center for Education Statistics, 2001.

[28] American Diploma Project. *Do Graduation Tests Measure Up*? Washington, DC: Achieve, Inc., 2004.

requirements for admission are less rigorous than the formally stated (but rarely enforced) expectations for a high school diploma. Almost never do they meet international standards for the transition from secondary to tertiary education.

Once admitted, even well-prepared students often score poorly on mathematics placement exams that demand high performance on a narrow range of skills peculiar to certain entry level mathematics courses. Broader mathematical and quantitative skills important for other college courses (and included in the panel's view of preparedness) are rarely acknowledged or rewarded in placement exams. In this way high school graduation requirements, postsecondary admissions requirements, and college placement exams form a three-ring cacophony of conflicting signals that spread confusion about what it really means to be mathematically prepared for college. To be effective, a preparedness-focused NAEP must reflect this broad range of priorities.

In 2003 Education Trust conducted a detailed comparison of NAEP with a sample of widely used college placement tests for the purpose of determining the extent to which the 12th Grade NAEP provides information about the readiness of high school seniors for credit-bearing mathematics courses.[29] This study found "no agreement" about what knowledge and skills define college readiness. Instead, it focused on readiness for college algebra as determined by widely used placement tests that have been validated for that purpose. It found that NAEP mathematics items are presented in a format so unlike those of placement exams as to "make them nearly incomparable."

The Education Trust study reveals that placement tests use words as little as possible: they lack both contextual or word problems as well as constructed response items. They focus much more heavily (sometimes exclusively) on algebra, and include items that require a higher degree of algebraic proficiency than does NAEP. In contrast, NAEP attempts to measure the success of the mathematics curriculum "at its broadest." Because of these differences, NAEP cannot (and should not) serve as a substitute for these placement exams.

***Issue 3**: Should NAEP use a single scale for mathematics?*

In item response theory (IRT), a *scale* is a set of values defined over a subset of items characterized by an underlying single-dimensional construct. It provides information about what students know and can do in a particular area. NAEP currently creates separate scales for each of its four 12th grade content areas in mathematics (number, algebra, data analysis and statistics, measurement and geometry). It then creates a composite scale for mathematics— which is what the public generally hears about— by means of a weighted average based on the relative importance assigned to each content strand.

Item response theory assumes that all test items belonging to a single content area require similar knowledge and skills that represent a single "latent trait." This assumption makes items commensurable, thus enabling them to be placed on a single scale. (The number of parameters characterizing the items depends on which particular latent trait model is employed; in NAEP the 3-parameter logistic model is used.) Item difficulties are estimated from the students' responses,

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29 "What can Grade 12 NAEP tell us about readiness for college-level work?" Education Trust, Sept 22, 2003.

as are the other parameters. The scale for each content area is then estimated from these item parameter estimates.

As NAEP may seek to expand the nature of assessment items to reflect a broader preparedness goal, the IRT assumption that all items (on a single scale) be based on common skills (or a single latent trait) is a cause for concern. One mathematician who has examined the special difficulties of applying IRT to mathematics assessments argues that scaling is reliable only when assessing standard procedural skills. "The more thoughtful, thus unpredictable, a test, the more likely it is that equating methods will misperform."[30] Others argue more generally that IRT methods are plausible only when the range of skills is narrow and well agreed by all experts[31]—a circumstance very different from NAEP's new goal of reporting on preparedness for diverse postsecondary endeavors.

The panel discussed extensively whether it is desirable to combine all of the mathematics content areas into a single scale. This issue is relevant to the question of defining preparedness because, hypothetically, the balance of importance among strands of mathematics may be different for someone entering college than for someone entering the world of work. Were that true, then it would be important to preserve different scales in order that achievement profiles could accurately represent students' preparedness for different kinds of postsecondary pursuits.

Evidence from previous NAEP administrations shows a high degree of inter-correlation among the different content scales in mathematics. Moreover, as the ADP study emphasized, there is virtually no separation of disciplines or topics in the work place; increasing emphases on interdisciplinary approaches make this apply also in undergraduate education.[32] Some infer from these trends that there may be little value to strand-specific scales (e.g., algebra, geometry) for a revised NAEP that is focused on workplace and college preparedness. Indeed, separate scales may send misleading messages about the true nature of preparedness (not to mention of mathematics).

Notwithstanding these minor drawbacks, many lines of reasoning support the value of retaining multiple scales. First, current high inter-correlations may be misleading or transient. They could be the result of ineffective instruction, leaving students to rely more on mathematical abilities than on learned skills. In this case, as instruction improves over time the observed correlations may decrease. They may also decrease from previous levels if NAEP introduces more challenging items appropriate to its new preparedness mission. Moreover, since the kinds of knowledge and skills required to solve problems in, say, data analysis and algebra are quite different, varied patterns of instruction may lead to very different performance profiles. These possibilities would render suspect any attempt to put all items on a single scale.

Another worry is that because of the way assessments like NAEP are constructed, items that demonstrate high discrimination—those that most clearly separate lower from higher ability students—come to dominate the pool of items from which the test forms are constructed. If all the items are on a single scale, this will inevitably tilt the pool of test items to the benefit of test-savvy, academically oriented students and lead to a test that is not particularly sensitive to the

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[30] Alan Tucker, "In Testing, How Reliable are Year-to-Year Comparisons?" *Education Week*, August 11, 2004.
[31] Paul Barton. *Unfinished Business*. Princeton, NJ: Educational Testing Service, Jan. 2005, pp. 20-25.
[32] *Greater Expectations*. Washington, DC: Association of American Colleges and Universities, 2002.

mathematics curriculum.

There also remain sound theoretical reasons for continuing to use separate scales. In order for items to be properly ordered by degree of difficulty, IRT theory assumes that test items require similar knowledge and skills. Psychometric considerations always favor items that are derived from a common domain. This assumption is more readily satisfied within narrow content areas (algebra, or data) than with mathematics items from many domains mixed in one single scale.

Indeed, an important reason for retaining (and reporting on) separate scales is that the information provided by such scales will be of benefit to educators in determining areas of strengths and weakness. Coupled with the Commission's recommendation to report state-level scores, content-oriented profiles of student performance would provide a powerful lever for improvement. Different scales for different mathematical competencies are very useful to be able to identify gaps in skills more precisely.

The major *potential* argument against separate content scales is the need to include in a revised NAEP a significant proportion of cross-cutting items that, by design, do not fit well into any of the four content categories. Both ADP and Achieve's earlier Mathematics Achievement Partnership (MAP) project stress the importance of such cross-cutting problems that require skills from more than one content domain. Panelists strongly agreed that it would be important to significantly increase the proportion of cross-cutting items on the 12th grade NAEP mathematics assessment—both for mathematical reasons and because this is one of the most important changes that can help reorient NAEP in the direction of preparedness for postsecondary endeavors where distinctions among content areas are less important. Since real life problems nearly always come in an integrated form (rarely as number, *or* algebra, *or* geometry, *or* data), it is essential to have many cross-cutting items on the new 12th grade NAEP.

The fact that mathematics is integrated virtually everywhere except in mathematics classes and that a revised NAEP would have many problems that reflect this characteristic does not render pointless separate scales. It does, however suggest the need for a separate scale to measure integrative ability. There are many different ways in which this can be done. Perhaps the best is to add a separate cross-cutting scale to the four current content areas, in part to ensure that test developers maintain an appropriate priority on problems of this type.

***Resolution*:** The 12th grade NAEP mathematics assessment should continue to use separate scales, but should add a new scale to measure cross-cutting performance.

Recommendation: NAEP should define preparedness for college, work, and the military to include (a) the mathematics required to qualify without remediation for credit-bearing courses in postsecondary education; (b) quantitative tools associated with data, computers, and probability; (c) broad competence in mathematical reasoning; and (d) ability to integrate multiple content areas in diverse contexts. The 12th grade NAEP mathematics assessment should be scored and reported on five separate scales— number properties and operations, measurement and geometry, data analysis and probability, algebra, and cross-cutting— that are subsequently combined into a single mathematics scale.

A related perspective can be found in OECD's Programme for International Student Assessment (PISA) whose purpose is very close to the proposed preparedness goal of NAEP.[33] PISA is an international test designed to gauge the capacity of 15-year-old students to engage with mathematics in ways that meet their needs as "constructive, concerned, and productive citizens." PISA emphasizes problems that are not limited to a single content area, that students can identify as being important, and that are not immediately solved through a studied process. The proposed cross-cutting scale for NAEP would share many characteristics of the PISA assessment. Comparative analyses may thus be productive and informative.

Question 2. What changes in mathematics objectives are necessary in order for NAEP to reflect the ADP expectations for high-trajectory postsecondary endeavors?

Although they differ in some topics and many details, ADP and the 2005 12th grade NAEP Framework agree on many major mathematics strands and objectives.[34] Differences arise chiefly from their different purposes: ADP attempts to articulate knowledge and skills that should be required of all high school graduates to be ready for productive postsecondary endeavors, whereas the 2005 NAEP Framework attempts to organize the domains of knowledge and skills that are currently included in typical high school mathematics curricula. Thus, for example, the 2005 NAEP Framework includes several relatively advanced topics that are not listed in the ADP benchmarks but that are commonly studied by those students who take four years of secondary school mathematics (e.g., arithmetic and geometric progressions, vector representation of velocity, effect of outliers on common statistical measures).

Doing Algebra for a Purpose

Suppose that Z = (X+Y)/XY, where X is constant and Y is increasing. What is happening to Z? Well, both the numerator and the denominator of Z are increasing, but it is not clear which one is doing it faster, or what is happening to the ratio. However, by rewriting Z as (1/X) + (1/Y), it is easier to see that is happening. One form of Z serves our purpose—to find out what is happening—whereas the other does not.

Similarly, some benchmarks in ADP are either not mentioned in the NAEP Framework or are not stressed to the degree that they are in ADP. Examples include complex numbers, geometric constructions, right triangle trigonometry, and statistical distinctions (correlation vs. causation, randomized vs. observational studies). Following recommendations for Grade 8 of Achieve's Mathematics Achievement Partnership (MAP)[35], ADP stresses the importance of computing with rational numbers "fluently and without a calculator" whereas in NAEP fractions only "supplement" decimal numbers when it is necessary to "represent ratios of two whole numbers exactly."[36] A few advanced (asterisked) topics in ADP are included in the NAEP framework (e.g., domain and range of functions), but most are not (e.g., rational exponents, arithmetic and

33 Programme for International Student Assessment. *Measuring Student Knowledge and Skills*. Paris: Organization for Economic Cooperation and Development, 2000.

34 "ADP Benchmarks and NAEP Mathematics Framework 2005: Side-by-Side Chart." Achieve, Inc. Internal Paper. December 15, 2004.

35 Mathematics Achievement Partnership. *Foundations for Success: Mathematics Expectations for the Middle Grades*. Washington, DC: Achieve, Inc., 2002.

36 Mathematics Framework, op. cit., p. 14.

geometric sequences, function composition and inverses, binomial theorem, and trigonometric functions).

Both ADP and NAEP have a common characteristic that is somewhat inconsistent with their broad purposes. Inventories of objectives and items reveal that some topics (e.g., number and measurement) appear primarily within the Basic region of the NAEP scale while others (notably algebra) are primarily Advanced. This leads to a profile at the 12th grade that is badly out of synch with the way students will be expected to use mathematics because some important topics (e.g., numbers, fractions, ratios, indirect measurement) do not advance much beyond 8th grade sophistication. It also fails to do justice to the connectedness and sophistication of mathematics itself. Better balance in which all areas of mathematics were assessed at all achievement levels would better serve NAEP's proposed new goal.

Reading Algebra with Meaning

One of the workplace tasks in Ready or Not considers two different actions of mixing fertilizer with water. In one, 5 gallons of fertilizer is mixed with 50 gallons of water; in the other, 10 gallons of fertilizer is mixed with 50 gallons of water. Is the second mixture twice as concentrated as the first? Using F for fertilizer and W for water, this leads to a comparison of two algebraic expressions. The first, 2(F/(F+W)), represents twice the concentration of the first mixture. The second, 2F/(2F+W), represents the concentration of the second mixture.

Students who can read algebra should see the difference between these two expressions in two ways. First, the expressions are *not equivalent* because the rules of algebra do not permit a transformation from one into the other. Second, there is a *difference in meaning* between the expressions: one means doubling the amount F of fertilizer added to the amount W of water and then calculating the concentration, whereas the other means doubling the concentration you would get by just adding F to W. The key is to be able to do both of these things, and therefore discover the interesting fact that doubling the amount of chemical added does not double the concentration.

The American Diploma Project documents how the worlds of work and higher education build on a single unified platform of important mathematical skills and understandings. ADP thus brings together two strands of research that have typically been conducted independently. One involves efforts to match school outcomes with the needs of work and the military, and includes such efforts as the SCANS report *What Work Requires of Schools*, the Armed Services Vocational Aptitude Battery (ASVAB), and ACT's *WorkKeys*. The other, more widely known, involves projections of college success based on college entrance exams such as SAT and ACT, as well as the "fractured" and largely invisible system of college placement exams.

Although the ADP report *Ready or Not* contains lists of mathematical benchmarks similar to those in scores of other standards documents, it alone anchors these standards in a series of actual college assignments and work place tasks. The standard of postsecondary preparedness set by ADP can be seen more vividly in these tasks than in a bare list of topics or expectations. Here, for example, one discovers the importance not just of doing algebra, but of reading algebra with insight, not just of manipulating symbols, but of calculating for a purpose. (See the two sidebars for examples of this distinction.)

The ADP tasks reveal that interpreting mathematics is very different from performing mathematics, and much more important. It is not an easy task for test writers to shift cognitive gears from procedures to insights, but it is a necessary change for NAEP if it is to achieve its new purpose of reporting on student preparedness for college, work, and the military. Some samples of mathematical tasks that illustrate the kind of purposeful and thoughtful mathematics that a revised

NAEP should assess are contained in Appendix I. (Examples in Appendix I are not intended to illustrate actual or potential NAEP items, but rather the kinds of mathematical thinking that NAEP items should seek to assess.)

Recommendation. To meet its new goals, the 12th grade NAEP Mathematics Framework should be revised and expanded to incorporate objectives derived from the ADP curricular benchmarks, reasoning expectations, workplace tasks, and sample assignments. In particular, NAEP should add a special emphasis on cross-cutting, purposeful problems such as those illustrated in the ADP workplace tasks.

Some suggestions for how this recommendation would affect the 12th grade NAEP mathematics objectives can be found in Appendix G. Appendix H illustrates what these new objectives might look like if these suggestions were fully implemented. (Note: Appendix G represents only the first draft of these recommendations. Extensive revisions based on further review have been incorporated in Appendix H, but not in Appendix G. Thus while Appendix G accurately illustrates the types of changes being recommended, the complete presentation of proposed revised objectives appears only in Appendix H.)

Question 3. What structural changes should NAEP consider to better reflect a new goal of reporting preparedness for postsecondary endeavors?

Issue 4: *What should be the extent and focus of levels of mathematical skills assessed by a revised 12 Grade NAEP?*

Currently the 12th grade NAEP Framework describes domains of mathematical knowledge commonly taught in secondary school. As its purpose shifts to a focus on preparedness for work and college, the Framework should expand to include items important for this goal that may not now be present, but it should not eliminate commonly taught topics just because they may not be absolutely necessary to meet a preparedness test for college and work. As noted earlier, objectives from common post-Algebra II courses such as AP Statistics and Precalculus are legitimate elements of the12th grade NAEP because some students in the NAEP sample can function at this mathematical level.

Similarly, since a relatively large proportion of 12th grade students are several years behind in mathematical proficiency, a parallel issue arises concerning elementary or "remedial" topics. If NAEP is to report on the mathematical proficiency of 12th grade students, it is important that the 12th grade Framework also include a good representation of items that may have been last taught in grades 6-8 but that are nonetheless important for all high school graduates. Indeed, deficiencies in graduates' performance on topics normally taught in these grades, especially from Number and Data, are cited more often than deficiencies in topics from Algebra II by employers and faculty who judge the capabilities of recent high school graduates.

By sampling only 12th grade students, NAEP misses over one-quarter of the 17-to-18-year-old population who have dropped out of school and may already be in the work force (or unemployed). One might plausibly assume that most students in this population would also have mathematical proficiencies approximating middle grade skills rather than secondary school standards. It is not unreasonable to conjecture, therefore, that approximately half of the young adults at the age for college, work, or military pursuits would display a range of mathematical skills more appropriate to the 8th grade NAEP than to the current 12th grade test. This poses a special challenge if the purpose of NAEP shifts from assessing 12th grade performance to preparation for work and college.

If NAEP is to help the nation monitor improvements in preparedness, it needs some means of providing a complete and accurate portrait. One option would be to extend its sample to the entire population of 17-to-18-year-olds. Alternatively, NAEP could continue its current practice of sampling only in schools, but then also report what percentage of 17-to-18-year-olds are represented by its sample.

These considerations pose an additional serious dilemma. On the one hand, the purpose of the revised NAEP is to report the mathematical preparation of our youth for college, work, and the military. On the other hand, current data suggests that the skills of many individuals do not match the expectations of career-track jobs or college programs. An assessment focused too narrowly on the preparedness goal would do a poor job of capturing the actual skill levels of many test-takers, leading no doubt to disaffection with the assessment.

***Resolution*:** The level of items on the revised 12th grade NAEP should span content from the 8th grade NAEP framework through precalculus and statistics (but not calculus). Selected items at the elementary and advanced levels that lie outside the central focus of preparedness expectations should be designated for light sampling that is, however, sufficient to produce reliable estimates of students' performance at these levels.

For purposes of discussion, we call these peripheral categories *brackets*. Items in the high bracket are optional for students (in that they are not essential to meet a preparedness goal) but are not optional for NAEP (since it is important that the nation know what proportion of 12th graders are proficient at this level). Items in the low bracket have a different character. They are so essential and so often lacking among high school graduates that even though they are included in the 8th grade NAEP framework the panel believes that a light sampling again at 12th grade is important. Appendices G and H identify certain objectives as either high or low bracket.

This expanded domain will enable NAEP to monitor a population that includes many students who are significantly underprepared for college, as well as many whose preparation exceeds the minimum required to succeed in credit-bearing courses. It will also enable NAEP to report on population characteristics in relation to the varied entrance expectations of different colleges and universities and different types of work or career settings.

One problem that will need to be addressed is the difficulty of getting a sufficient sample of elementary items. Here NAEP's goal may clash with statistical standards. By stressing preparedness for high-performance work and credit-bearing college courses, NAEP will be sampling most heavily at the least dense end of the performance spectrum. Consequently, as the WorkKeys study cited above documented, NAEP will provide the least information about the mathematical skills of the majority of students.

Issue 5: *What range and dimensions of mathematical skills should be assessed by a revised 12th Grade NAEP?*

The current NAEP Framework outlines a weighting of items along several dimensions: content, complexity, response type, and calculator use. As NAEP shifts to a preparedness goal, some of these categories and weights should shift to reflect the broader purpose of the revised assessment.

Content. In reviewing the ADP mathematics benchmarks and the 2005 NAEP mathematics objectives, panelists identified certain areas of mathematics that were insufficiently represented in the 2005 NAEP framework (e.g., logic, sets, binary numbers, non-geometric measurement). These omissions can easily be rectified; some suggestions for how to do so are contained in Appendix H. More worrisome is a noticeable emphasis on the style and tone of classroom mathematics that is insufficiently balanced with language that reflects the way mathematics appears outside mathematics classrooms. This is more difficult to correct, but very important if NAEP is to effectively assess mathematical preparedness for endeavors other than just the study of more mathematics.

As noted above, one important content change implied by a change in the purpose of NAEP is to add problems that cut across the different mathematical domains. Reporting student performance that reflects this change will require the use of five separate scales (rather than the four currently used). Panelists recommend that these scales reflect the following distribution:

10%	Number
20%	Geometry
20%	Data Analysis
25%	Algebra
25%	Cross-Cutting
100%	Total

The new cross-cutting scale might provide real insight into a dimension of student capability that employers and faculty often emphasize. The single composite "mathematics" scale used for broad public reporting should be created by combining these five scales based on the weighting given above.

The panel also recognizes the importance of continuity with previous NAEP assessments, especially as the nation monitors changes in performance. For such purposes, we note that the cross-cutting items can be assigned, however imperfectly, to primary content areas in order to enable an analysis that employs the four-scale system used in previous NAEP assessments. To ensure appropriate balance, we further recommend that the primary content assignments of the 25% of items classified as cross cutting be divided as follows: 10% geometry, 5% data analysis, and 10% algebra. In other words, 40% of the items classified as cross-cutting should be able to be classified primarily as geometry, 20% as data analysis, and 40% as algebra. Doing this will yield the following weights when the data is analyzed in four content areas: 10% Number, 30% Geometry, 25% Data Analysis, and 35% Algebra--the same distribution as in the 2005 NAEP assessment.

Note that some changes are suggested in the names of the four content areas. This is because the topics included in these domains are expanded and revised in Appendices G and H (as explained elsewhere). In addition, the new goal for NAEP at the 12th grade level lifts the rationale for maintaining 4th and 8th grade categories in the 12th grade NAEP. Very broad names for the content areas provide a more flexible framework.

***Recommendation*:** A preparedness-focused NAEP should classify items and report results using five scales—four domain-specific and one cross-cutting—using the weights given above. For purposes of public reporting, these five scales should be combined by means of a weighted average to produce a single mathematics scale.

***Complexity*.** The 2005 NAEP defines three complexity levels (low, moderate, high; see Appendix D) and recommends that half of the total possible score be based on items of moderate complexity, with the remainder based equally on items of low and high complexity. As noted above, this 2005 organization is a sequel to an earlier NAEP effort that stressed students' mathematical "abilities" and "power." The 2005 complexity dimension focuses instead on characteristics of items.

The NAEP approach to complexity is neither inevitable nor unique. Its own history shows the ephemeral character of these classifications. Others who have thought seriously about mathematical tasks have proposed different schemes. For example, Achieve's Mathematics Achievement Partnership (MAP) outlined five types of complexity:

- *Procedural.* Items that test the accurate use of practiced routines. In multiple choice items, distractors are designed to catch faulty manipulation.
- *Focused Conceptual.* Items that require understanding of one or two mathematical concepts. Distractors tend to be based on misconceptions.
- *Cross-Cutting Conceptual.* Like focused conceptual items, but drawing on concepts from several different areas of mathematics.
- *Strategic.* Items that pose a strategic challenge and require both procedural skill and conceptual understanding. Generally these are constructed response items.
- *Conjecture, Generalize, Prove.* Items that require the student to make, follow, or evaluate a mathematical argument.

Further, complexity is not the only way to think about the texture and degree of mathematical proficiency. An influential report from the National Research Council identifies five quite different aspects of mathematical proficiency:[37]

- *Conceptual understanding*. Comprehension of mathematical concepts, operations, and relations.
- *Procedural fluency*. Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
- *Strategic competence.* Ability to formulate, represent, and solve mathematical problems.
- *Adaptive reasoning.* Capacity for logical thought, reflection, explanation, and justification.
- *Productive disposition.* Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in the value of diligence and in one's own efficacy.

[37] Jeremy Kilpatrick, et al. *Adding It Up: Helping Children Learn Mathematics*. Mathematics Learning Study Committee, National Research Council. Washington, DC: National Academies Press, 2001.

The ADP study implicitly adds other potential dimensions:

- *Contextual.* Items that simulate societal, scientific, or workplace tasks where mathematical issues are embedded, sometimes hidden, in specific situations.
- *Interpretive.* Items that require insight and interpretation rather than solution.

Different types and levels of complexity arise in each of these different dimensions.

In light of the many equally plausible approaches to classifying complexity, the panel decided that it had no good grounds for recommending any significant change in the descriptions and balance described in the 2005 NAEP Framework. Only minor rewording seems warranted.

Recommendation: A revised NAEP should continue to use the same three levels of complexity that are part of the 2005 NAEP, and continue the 25%-50%-25% distribution pattern of the 2005 NAEP. Slightly revised descriptions of the complexity levels can be found in Appendix F.

***Response Type*.** In 2005 the 12th grade NAEP Framework specifies that approximately half of testing time be allocated to multiple-choice items, with the remaining half devoted to constructed response items, both short (35%) and extended (15%). Even though extended constructed response items are expensive both in student test time and grading, they are arguably the most important items in assessment since they alone give insight into students' thinking. The panel recognizes that multiple choice and short constructed response items can represent many if not most of the intended content and complexity dimensions of the NAEP assessment. Indeed, well designed distractors and tallies of common response errors can reveal patterns in student misconceptions that can be very helpful to guide instruction.

Yet by their very nature multiple choice and short response items are stereotypical of an artificial face of mathematics that is far removed from the characteristics sought by employers and educators. To ensure sound evidence of preparedness for college and work, NAEP must maintain its current average of 15% extended response items. Of the remaining 85%, the proportions represented by multiple choice and short constructed response—currently 50% and 35%, respectively—is of less importance and might be altered for practical or technical reasons.

***Recommendation*:** At least 15% of NAEP testing time should be devoted to extended constructed response items and no more than 50% of testing time should be devoted to multiple choice items. The remainder should be short constructed response items.

***Calculator Use*.** Use and abuse of calculators is widespread in school and society, as is concern about such use and abuse. Unfortunately, neither the current NAEP guidelines nor the ADP report address issues of the effective use of technology in a meaningful way. The current framework specifies that two-thirds of the 12th grade NAEP mathematics test should prohibit use of a calculator, while one-third should allow such use. Questions on the calculator portion of the test are to be designed with calculator use in mind while at the same time ensuring that no items provide an advantage to students with a graphing calculator. With few exceptions, neither the ADP mathematics expectations (Appendices A and B) nor the 2005 NAEP mathematics objectives (Appendix C) say anything about where and how calculators should be properly used.

Except in parts of some mathematics courses, the contemporary practice of mathematics in the workplace, the military, and in most college courses is mediated through powerful technology. Common examples include spreadsheets, graphing calculators, computer algebra systems, statistical packages, CAD/CAM programs, and dynamic geometry packages. Employers and college instructors expect students to be familiar with calculator and computer tools and to use these tools intelligently in the solution of mathematical problems. Schools therefore have an obligation to introduce students to the proper use of technological tools as an aspect of proficiency in mathematics, and a preparedness-focused NAEP has some obligation to assess the degree to which students are gaining in this important proficiency.

Clearly NAEP cannot directly assess students' ability to use computers. For socioeconomic reasons, it may also be inappropriate to assess directly all but the most rudimentary uses of calculators. However, item writers can devise questions that probe students' experience with and understanding of common computer or calculator operations and thereby assess the degree to which students are gaining these important experiences. It is not necessary for specialized technological tools to be used during the assessment to monitor students' capabilities in this important arena.

***Recommendation*:** A preparedness-focused NAEP Mathematics Framework should include objectives concerning the appropriate use of technology in the solution of mathematical and quantitative problems. The NAEP assessment should introduce multiple choice or short constructed response questions that probe students' experiences with common technological tools for mathematics.

Question 4. What changes should be considered to NAEP's achievement level descriptions to enable meaningful reporting of student preparedness?

In NAEP assessments, scales indicate *what is* while achievement levels indicate *what should be*. In 2005, as in earlier years, NAEP established three cut-points on the mathematics scale to distinguish four achievement levels: advanced, proficient, basic, and below basic. The panel discussed extensively whether these levels—their number, their names, and their descriptions—are suitable to NAEP's proposed preparedness focus. Potential differences in the mathematical demands of college, work, and the military complicate these concerns.

***Issue 6*.** *How can achievement levels effectively describe student preparedness?*

As we have already noted, different postsecondary options build on a broad common base of mathematical knowledge and skills. The more students know and are able to do, the greater their options for college programs and career paths. Increasing levels of preparedness expand students' choices just as concentric circles encompass increasing area centered on a common point. In a new preparedness-focused NAEP, the achievement levels should describe these concentric circles of preparation.

The primary achievement level that corresponds to what NAEP currently calls *proficient* should, as we have noted, be termed *prepared*. This is intended to signal adequate preparation for credit-bearing college courses and entry level positions in career-track jobs or the military. Many factors other than mathematical proficiency contribute to students' success in college or careers, and the NAEP mathematics assessment cannot address these other factors. What it can report, however, and what it should be seen as reporting, is the proportion of students who are prepared, assuming other factors are favorable, to succeed in entry level credit-bearing postsecondary courses and equivalent training for high-growth jobs or military careers.

Below this prepared level NAEP can report the proportion of students who are *partially prepared*, perhaps with two tiers that correspond approximately to the degree of remediation needed to become prepared for credit-bearing courses or entry-level jobs. Above prepared there should be an advanced level called *well prepared* that corresponds approximately to the proportion of students who will leave high school prepared to take calculus or other comparable college-level mathematics courses.

It is important to note that neither the current nor a revised 12th grade NAEP can vouch fully for students' preparation for calculus. Many of the prerequisite skills are not even included in the NAEP framework. However, because of the highly structured nature of secondary school mathematics, it is likely that a high correlation can be established between certain levels on NAEP and students' preparedness for calculus.

Recommendation: Achievement levels in a preparedness-focused NAEP should center on the category of *prepared* (instead of *proficient*) where prepared is defined as preparation for credit bearing college courses or for entry-level employment in career-track jobs and corresponding military positions. Below prepared should be a category of *partially prepared* that signifies the need for only one remedial mathematics course. Above prepared would be a level called *well prepared* signifying the proportion of students who may be prepared for calculus or comparable courses.

Those who are not partially prepared would need two or more remedial courses before being ready for credit-bearing college work in mathematics. Among those who score at the well-prepared level would be some who are not quite ready for calculus, some who are fully prepared for calculus, and some who have already completed calculus. As noted above, NAEP is incapable of making distinctions among students at this level of detail.

Issue 7: *How should a revised NAEP report the general level of mathematical preparedness for postsecondary options?*

This issue generated considerable discussion with panelists views evolving as different issues emerged. On the one hand are arguments for specificity, for reporting different scales aligned with different mathematical priorities of college, work, and the military. Some wondered further whether the college scale should be either extended or separated in order to assess students' preparation for mathematically intensive programs (which are both different from and more extensive than the preparation required for many liberal arts majors).

On the other hand are strong arguments for unity, for a single preparedness expectation for all postsecondary options. This view is consistently expressed by ADP and Achieve in other contexts, primarily to ensure that students leave high school with a multiplicity of options still open. Providing choices among postsecondary options requires adequate preparation for college *and* work *and* military, even when mathematical priorities among these options may differ in certain details.

Since many items from all content domains of mathematics are highly relevant to all three trajectories, potential differentiated scales for postsecondary choices would necessarily have a high degree of overlap—most likely a greater overlap than, say, the algebra scale has with the geometry scale. Any claim that a proposed scale of preparedness for college might be significantly different from one for work would be very difficult to validate. Indeed, as noted above, most career-trajectory jobs (and comparable military assignments) require some type of training beyond high school.

Panelists resolved these conflicting objectives by recommending two types and levels of reporting. One, intended primarily for the broad public, would be a unified summary that reports the proportion of students who are prepared *across all five content scales for all postsecondary options*. The other, intended for education and policy experts, would be a detailed analysis that provides focused information for instructional and policy decisions. (This version is discussed further under Issue 8 below).

This recommendation sets a high bar for preparedness. It says that strong preparation in data cannot make up for weak preparation in algebra, or vice versa. It says that to be fully prepared, students leaving high school must be prepared for all three options--college, work, and military. It says that a high school diploma should signal the achievement of recognized educational objectives regardless of what a student will do next.

***Recommendation**:* Based on the desire to preserve students' postsecondary options, the extensive mathematical overlap in potential college-work-military scales, and the common need for postsecondary education for nearly every career option, NAEP should report only one set of achievement levels on its unified mathematics report. These levels should signify preparedness across all five NAEP scales for all three postsecondary options.

***Issue 8**. How should a revised NAEP report specific preparedness levels associated with different postsecondary objectives?*

For several different reasons—some mathematical, some statistical, some sociological—the panel is strongly opposed to any reporting system that locates different achievement levels (cut points) for college, work, and the military on the unified mathematics scale. As noted earlier, the mathematical needs of these areas overlap greatly and cannot be reliably distinguished by available psychometric tools. Moreover, achievement level cut points located on this single scale might result in an implied hierarchy of postsecondary options that will only reinforce class distinctions.

However, the panel recognizes that different postsecondary options may well have different priorities in regards to the five proposed NAEP scales—four in content domains, one cross-cutting. For example, the level of algebra necessary to be prepared for college may be higher than the level required for work; the military may place greater demands on cross-cutting expertise; etc. Thus in addition to setting achievement levels and associated cut-points on the unified mathematics scale, the panel also thought it would be important for results to be reported at a fine enough grain-size that differences in *priorities* among the three postsecondary aspirations might be communicated.

To permit reliable analysis at this level, each scale needs to include a full range of item complexity. The panel was concerned that the current NAEP mathematics achievement levels unwisely correspond to a sequential hierarchy of content areas (e.g., below basic is predominantly number, basic is largely measurement and geometry, proficient is dominated by data and probability, and advanced is largely algebra). This correlation of content with level is flawed in two respects: First, every domain of mathematics should be examined at all levels of proficiency. Second, by implicitly linking content to level, students could be counted as proficient with very little or no algebra. A revised NAEP needs to ensure that each of its five scales include items that permit reliable association with each level of preparedness.

Thus, at the most detailed level of analysis, experts representing each of the three postsecondary outcomes—college, work, and the military—should prepare descriptions of what students should be able to do in each of the five content scales at each achievement level. This would yield a fifteen cell matrix that would characterize the nuances of preparedness at each achievement level for different postsecondary options. Although more complicated than the current system, this process would yield the kinds of specific information that states need to focus their efforts on improving instruction to prepare students for an array of postsecondary options.

Resolution: Achievement level expectations for different postsecondary options should be developed and reported separately on NAEP's five scales rather than as different cut points on NAEP's composite mathematics scale.

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Appendix A: American Diploma Project Mathematics Benchmarks

Because major areas of study at postsecondary institutions have different prerequisites, certain mathematics benchmarks are marked with an asterisk (*). These asterisked benchmarks represent content that is recommended for all students, but is required for those students who plan to take calculus in college, a requisite for mathematics and many mathematics-intensive majors.

I. Number Sense and Numerical Operations

The high school graduate can:

I1. Compute with rational numbers fluently and accurately without a calculator:

- I1.1. Add, subtract, multiply and divide integers, fractions and decimals.
- I1.2. Calculate and apply ratios, proportions, rates and percentages to solve problems.
- I1.3. Use the correct order of operations to evaluate arithmetic expressions, including those containing parentheses.
- I1.4. Explain and apply basic number theory concepts such as prime number, factor, divisibility, least common multiple and greatest common divisor.
- I1.5. Multiply and divide numbers expressed in scientific notation.

I2. Recognize and apply magnitude (absolute value) and ordering of real numbers:

- I2.1. Locate the position of a number on the number line, know that its distance from the origin is its absolute value and know that the distance between two numbers on the number line is the absolute value of their difference.
- I2.2. Determine the relative position on the number line of numbers and the relative magnitude of numbers expressed in fractional form, in decimal form, as roots or in scientific notation.

I3. Understand that to solve certain problems and equations, number systems need to be extended from whole numbers to the set of all integers (positive, negative and zero), from integers to rational numbers, from rational numbers to real numbers (rational and irrational numbers) and from real numbers to complex numbers; define and give examples of each of these types of numbers.

I4. Understand the capabilities and the limitations of calculators and computers in solving problems:

- I4.1. Use calculators appropriately and make estimations without a calculator regularly to detect potential errors.
- I4.2. Use graphing calculators and computer spreadsheets.

J. Algebra

The high school graduate can:

J1. Perform basic operations on algebraic expressions fluently and accurately:

- J1.1. Understand the properties of integer exponents and roots and apply these properties to simplify algebraic expressions.
- J1.2. * Understand the properties of rational exponents and apply these properties to simplify algebraic expressions.
- J1.3. Add, subtract and multiply polynomials; divide a polynomial by a low degree polynomial.
- J1.4. Factor polynomials by removing the greatest common factor; factor quadratic polynomials.
- J1.5. Add, subtract, multiply, divide and simplify rational expressions.
- J1.6. Evaluate polynomial and rational expressions and expressions containing radicals and absolute values at specified values of their variables. J1.7. * Derive and use the formulas for the general term and summation of finite arithmetic and geometric series; find the sum of an infinite geometric series whose common ratio, r, is in the interval (-1, 1).

J2. Understand functions, their representations and their properties:

- J2.1. Recognize whether a relationship given in symbolic or graphical form is a function.
- J2.2. * Determine the domain of a function represented in either symbolic or graphical form.
- J2.3. Understand functional notation and evaluate a function at a specified point in its domain.
- J2.4. * Combine functions by composition, as well as by addition, subtraction, multiplication and division.
- J2.5. * Identify whether a function has an inverse and when functions are inverses of each other; explain why the graph of a function and its inverse are reflections of one another over the line y = x.
- J2.6. * Know that the inverse of an exponential function is a logarithm, prove basic properties of a logarithm using properties of its inverse and apply those properties to solve problems.

J3. Apply basic algebraic operations to solve equations and inequalities:

- J3.1. Solve linear equations and inequalities in one variable including those involving the absolute value of a linear function.
- J3.2. Solve an equation involving several variables for one variable in terms of the others.
- J3.3. Solve systems of two linear equations in two variables.
- J3.4. * Solve systems of three linear equations in three variables.
- J3.5. Solve quadratic equations in one variable.

J4. Graph a variety of equations and inequalities in two variables, demonstrate understanding of the relationships between the algebraic properties of an equation and the geometric properties of its graph, and interpret a graph:

- J4.1. Graph a linear equation and demonstrate that it has a constant rate of change.
- J4.2. Understand the relationship between the coefficients of a linear equation and the slope and x- and y-intercepts of its graph. (Associated Postsecondary Assignment: #3) J4.3. Understand the relationship between a solution of a system of two linear equations in two variables and the graphs of the corresponding lines. J4.4. Graph the solution set of a linear inequality and identify whether the solution set is an open or a closed half-plane; graph the solution set of a system of two or three linear inequalities.
- J4.5. Graph a quadratic function and understand the relationship between its real zeros and the x-intercepts of its graph.
- J4.6. * Graph ellipses and hyperbolas whose axes are parallel to the x and y axes and demonstrate

understanding of the relationship between their standard algebraic form and their graphical characteristics.

J4.7. Graph exponential functions and identify their key characteristics.

J4.8. Read information and draw conclusions from graphs; identify properties of a graph that provide useful information about the original problem.

J5. Solve problems by converting the verbal information given into an appropriate mathematical model involving equations or systems of equations; apply appropriate mathematical techniques to analyze these mathematical models; and interpret the solution obtained in written form using appropriate units of measurement:

J5.1. Recognize and solve problems that can be modeled using a linear equation in one variable, such as time/rate/distance problems, percentage increase or decrease problems, and ratio and proportion problems.

J5.2. Recognize and solve problems that can be modeled using a system of two equations in two variables, such as mixture problems.

J5.3. Recognize and solve problems that can be modeled using a quadratic equation, such as the motion of an object under the force of gravity.

J5.4. Recognize and solve problems that can be modeled using an exponential function, such as compound interest problems.

J5.5. * Recognize and solve problems that can be modeled using an exponential function but whose solution requires facility with logarithms, such as exponential growth and decay problems.

J5.6. Recognize and solve problems that can be modeled using a finite geometric series, such as home mortgage problems and other compound interest problems. (J6. * Understand the binomial theorem and its connections to combinatorics, Pascal's triangle and probability.

K. Geometry

The high school graduate can:

K1. Understand the different roles played by axioms, definitions and theorems in the logical structure of mathematics, especially in geometry:

K1.1. Identify, explain the necessity of and give examples of definitions, axioms and theorems.

K1.2. State and prove key basic theorems in geometry such as the Pythagorean theorem, the sum of the angles of a triangle is 180 degrees, and the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

K1.3. Recognize that there are geometries, other than Euclidean geometry, in which the parallel postulate is not true.

K2. Identify and apply the definitions related to lines and angles and use them to prove theorems in (Euclidean) geometry, solve problems, and perform basic geometric constructions using a straight edge and compass:

K2.1. Identify and apply properties of and theorems about parallel lines and use them to prove theorems such as two lines parallel to a third are parallel to each other and to perform constructions such as a line parallel to a given line through a point not on the line.

K2.2. Identify and apply properties of and theorems about perpendicular lines and use them to prove theorems such as the perpendicular bisectors of line segments are the set of all points equidistant from the two end points and to perform constructions such as the perpendicular bisector of a line segment.

K2.3. Identify and apply properties of and theorems about angles and use them to prove theorems such as two lines are parallel exactly when the alternate interior angles they make with a transversal are equal and to perform constructions such as the bisector of an angle.

K3. Know the basic theorems about congruent and similar triangles and use them to prove additional theorems and solve problems.

K4. Know the definitions and basic properties of a circle and use them to prove basic theorems and solve problems.

K5. Apply the Pythagorean theorem, its converse and properties of special right triangles to solve problems.

K6. Use rigid motions (compositions of reflections, translations and rotations) to determine whether two geometric figures are congruent and to create and analyze geometric designs.

K7. Know about the similarity of figures and use the scale factor to solve problems.

K8. Know that geometric measurements (length, area, perimeter, volume) depend on the choice of a unit and that measurements made on physical objects are approximations; calculate the measurements of common plane and solid geometric figures:

K8.1. Understand that numerical values associated with measurements of physical quantities must be assigned units of measurement or dimensions; apply such units correctly in expressions, equations and problem solutions that involve measurements; and convert a measurement using one unit of measurement to another unit of measurement.

K8.2. Determine the perimeter of a polygon and the circumference of a circle; the area of a rectangle, a circle, a triangle and a polygon with more than four sides by decomposing it into triangles; the surface area of a prism, a pyramid, a cone and a sphere; and the volume of a rectangular box, a prism, a pyramid, a cone and a sphere.

K8.3. Know that the effect of a scale factor k on length, area and volume is to multiply each by k, k^2 and k^3, respectively.

K9. Visualize solids and surfaces in three-dimensional space when given two-dimensional representations (e.g., nets, multiple views) and create two-dimensional representations for the surfaces of three-dimensional objects.

K10. Represent geometric objects and figures algebraically using coordinates; use algebra to solve geometric problems:

K10.1. Express the intuitive concept of the "slant" of a line in terms of the precise concept of slope, use the coordinates of two points on a line to define its slope, and use slope to express the parallelism and perpendicularity of lines.

K10.2. Describe a line by a linear equation.

K10.3. Find the distance between two points using their coordinates and the Pythagorean theorem.

K10.4.* Find an equation of a circle given its center and radius and, given an equation of a circle, find its center and radius.

K11. Understand basic right-triangle trigonometry and apply it to solve problems:

K11.1. Understand how similarity of right triangles allows the trigonometric functions sine, cosine and tangent to be defined as ratios of sides and be able to use these functions to solve problems.

K11.2. Apply the trigonometric functions sine, cosine and tangent to solve for an unknown length of a side of a right triangle, given one of the acute angles and the length of another side.

K11.3. Use the standard formula for the area of a triangle, A = 1/2bh, to explain the area formula, A = 1/2absinC where a and b are the lengths of two sides of a triangle and C is the measure of the included angle formed by these two sides, and use it to find the area of a triangle when given the lengths of two of its sides and the included angle.

K12. Know how the trigonometric functions can be extended to periodic functions on the real line, derive basic formulas involving these functions, and use these functions and formulas to solve problems:*

K12.1.* Know that the trigonometric functions sine and cosine, and thus all trigonometric functions, can be extended to periodic functions on the real line by defining them as functions on the unit circle, that radian measure of an angle between 0 and 360 degrees is the arc length of the unit circle subtended by that central angle, and that by similarity, the arc length s of a circle of radius r subtended by a central angle of measure t radians is s = rt.

K12.2.* Know and use the basic identities, such as $\sin^2(x) + \cos^2(x) = 1$ and $\cos(\pi/2-x) = \sin(x)$ and formulas for sine and cosine, such as addition and double angle formulas.

K12.3.* Graph sine, cosine and tangent as well as their reciprocals, secant, cosecant and cotangent; identify key characteristics.

K12.4.* Know and use the law of cosines and the law of sines to find missing sides and angles of a triangle.

L. Data Interpretation, Statistics & Probability

The high school graduate can:

L1. Explain and apply quantitative information:

L1.1. Organize and display data using appropriate methods (including spreadsheets) to detect patterns and departures from patterns.

L1.2. Read and interpret tables, charts and graphs.

L1.3. Compute and explain summary statistics for distributions of data including measures of center (mean, median) and spread (range, percentiles, variance, standard deviation).

L1.4. Compare data sets using graphs and summary statistics.

L1.5. Create scatter plots, analyze patterns and describe relationships in paired data.

L1.6. Know the characteristics of the Gaussian normal distribution (bell shaped curve).

L2. Explain and critique alternative ways of presenting and using information:

L2.1. Evaluate reports based on data published in the media by considering the source of the data, the design of the study, and the way the data are analyzed and displayed.

L2.2. Identify and explain misleading uses of data.

L2.3. Recognize when arguments based on data confuse correlation with causation.

L3. Explain the use of data and statistical thinking to draw inferences, make predictions and justify conclusions:

L3.1. Explain the impact of sampling methods, bias and the phrasing of questions asked in data collection and the conclusions that can rightfully be made.

L3.2. Design simple experiments or investigations to collect data to answer questions of interest.

L3.3. Explain the differences between randomized experiments and observational studies.

L3.4. Construct a scatter plot of a set of paired data, and if it demonstrates a linear trend, use a graphing calculator to find the regression line that best fits this data; recognize that the correlation coefficient measures goodness of fit; explain when it is appropriate to use the regression line to make predictions.

L4. Explain and apply probability concepts and calculate simple probabilities:

L4.1. Explain how probability quantifies the likelihood that an event occurs in terms of numbers.

L4.2. Explain how the relative frequency of a specified outcome of an event can be used to estimate the probability of the outcome.

L4.3. Explain how the law of large numbers can be applied in simple examples.

L4.4. Apply probability concepts such as conditional probability and independent events to calculate simple probabilities.

L4.5. Apply probability concepts to practical situations to make informed decisions.

Appendix B: ADP Expectations for Mathematical Reasoning

(The American Diploma Project. *Ready or Not: Creating a High School Diploma that Counts*. Achieve, Inc., 2004, p. 55.)

The study of mathematics is an exercise in reasoning. Beyond acquiring procedural mathematical skills with their clear methods and boundaries, students need to master the more subjective skills of reading, interpreting, representing and "mathematicizing" a problem. As college students and employees, high school graduates will need to use mathematics in contexts quite different from the high school classroom. They will need to make judgments about what problem needs to be solved and, therefore, about which operations and procedures to apply. Woven throughout the four domains of mathematics — Number Sense and Numerical Operations; Algebra; Geometry; and Data Interpretation, Statistics and Probability — are the following mathematical reasoning skills:

- Using inductive and deductive reasoning to arrive at valid conclusions.
- Using multiple representations to represent problems and solutions.
- Understanding the role of definitions, proofs and counter-examples in mathematical reasoning; constructing simple proofs.
- Using the special symbols of mathematics correctly and precisely.
- Recognizing when an estimate or approximation is more appropriate than an exact answer.
- Distinguishing relevant from irrelevant information, identifying missing information, and either finding what is needed or making appropriate estimates.
- Recognizing and using the process of mathematical modeling when mathematical structures are embedded in other contexts.
- When solving problems, thinking about strategy, testing ideas, trying different approaches, checking for errors and reasonableness of solutions, and devising ways to verify results.
- Shifting regularly between the specific and the general, using examples to understand general ideas, and extending specific results to more general cases to gain insight.

Appendix C: 2005 NAEP 12th Grade Mathematics Objectives

Number Properties and Operations

1. Number Sense

d) Write, rename, represent, or compare real numbers (e.g., π, $\sqrt{2}$, numerical relationships using number lines, models, or diagrams).

f) Represent very large or very small numbers using scientific notation in meaningful contexts.

g) Find or model absolute value or apply to problem situations.

h) Interpret calculator or computer displays of numbers given in scientific notation.

j) Order or compare real numbers, including very large or small real numbers.

2. Estimation

a) Establish or apply benchmarks for real numbers in contexts.

b) Make estimates of very large or very small numbers appropriate to a given situation by:

- identifying when estimation is appropriate or not,
- determining the level of accuracy needed,
- selecting the appropriate method of estimation, or
- analyzing the effect of an estimation method on the accuracy of results.

c) Verify solutions or determine the reasonableness of results in a variety of situations including scientific notation, calculator, and computer results.

d) Estimate square or cube roots of numbers less than 1,000 between two whole numbers.

3. Number Operations

a) Perform computations with real numbers including common irrational numbers or the absolute value of numbers. (*Include items that are not placed in a context and require computation with common and decimal fractions, as well as items that use a context*.)

d) Describe the effect of multiplying and dividing by numbers including the effect of multiplying or dividing a real number by: zero, a number less than zero, a number between zero and one, one, or a number greater than one.

g) Solve application problems involving numbers, including rational and common irrationals, using exact answers or estimates as appropriate.

4. Ratios and Proportional Reasoning

b) Use proportions to model problems.

c) Use proportional reasoning to solve problems (including rates).

d) Solve problems involving percentages (including percent increase and decrease, interest rates, tax, discount, tips, or part/whole relationships).

5. Properties of Number and Operations

b) Solve problems involving factors, multiples, or prime factorization.

c) Use prime or composite numbers to solve problems.

d) Use divisibility or remainders in problem settings.

e) Apply basic properties of operations. (*Properties include commutative, associative, and distributive properties of addition and multiplication*.)

f) Provide a mathematical argument about a numerical property or relationship.

Measurement

1. Measuring Physical Attributes

c) Estimate or compare perimeters or areas of two-dimensional geometric figures.

d) Estimate or compare volumes or surface area of three-dimensional figures.

e) Solve problems involving the coordinate plane such as the distance between two points, the midpoint of a segment, or slopes of perpendicular or parallel lines.

f) Solve problems of angle measure, including those involving triangles or other polygons or parallel lines cut by a transversal.

h) Solve mathematical or real-world problems involving perimeter or area of plane figures such as polygons, circles, or composite figures.

j) Solve problems involving volume or surface area of rectangular solids, cylinders, cones, pyramids, prisms, spheres, or composite shapes.

k) Solve problems involving indirect measurement such as finding the height of a building by finding the distance to the base of the building and the angle of elevation to the top.

l) Solve problems involving rates such as speed, density, population density, or flow rates.

m) Use trigonometric relations in right triangles to solve problems.

2. Systems of Measurement

a) Select or use appropriate type of unit for the attribute being measured such as volume or surface area.

b) Solve problems involving conversions within or between measurement systems, given the relationship between the units. (*Can include cubic units, and rates such as miles per hour to feet per second*.)

e) Determine appropriate accuracy of measurement in problem situations (e.g., the accuracy of measurement of the dimensions to obtain a specified accuracy of area) and find the measure to that degree of accuracy. (*For example, how accurately must we measure a piece of paper that is between 7 and 8 inches wide and 11 and 12 inches long to determine its area to 1/10 of a square inch accuracy?)*

f) Construct or solve problems (e.g., number of rolls needed for insulating a house) involving scale drawings.

g) Compare lengths, areas or volumes of similar figures using proportions.

Geometry

1. Dimension and Shape

b) Use two-dimensional representations of three-dimensional objects to visualize and solve problems involving surface area and volume. (*For example, show how doubling the radius of the bottom of a can affects the area of the label or the volume of the can*.)

c) Give precise mathematical descriptions or definitions of geometric shapes in the plane and in 3-dimensional space. (*Include full set of Platonic solids (e.g., cube, regular tetrahedron)*.)

d) Draw or sketch from a written description plane figures (e.g., isosceles triangles, regular polygons, curved figures) and planar images of 3- dimensional figures (e.g., polyhedra, spheres, and hemispheres).

e) Describe or analyze properties of spheres and hemispheres.

2. Transformation of Shapes and Preservation of Properties

a) Recognize or identify types of symmetries (e.g., point, line, rotational, self-congruences) of two- and three- dimensional figures.

b Give or recognize the precise mathematical relationship (e.g., congruence, similarity, orientation) between a figure and its image under a transformation.

c) Perform or describe the effect of a single transformation on two- and three-dimensional geometric shapes (reflections across lines of symmetry, rotations, translations, dilations).

d) Describe the final outcome of successive transformations.

e) Justify relationships of congruence and similarity, and apply these relationships using scaling and proportional reasoning.

3. Relationships between Geometric Figures

b) Apply geometric properties and relationships in solving multi-step problems in two and three dimensions (including rigid and non-rigid figures).

c) Represent problem situations with geometric models to solve mathematical or real- world problems.

d) Use the Pythagorean theorem to solve distance problems in two- or three-dimensional situations. (*Students are expected to recall the Pythagorean theorem.*)

e) Describe and analyze properties of circles (e.g., perpendicularity of tangent and radius, angle inscribed in a semicircle).

f) Analyze properties or relationships of triangles, quadrilaterals, and other polygonal plane figures.

g) Describe or analyze properties and relationships of parallel, perpendicular or intersecting lines including the angle relationships that arise in these cases.

4. Position and Direction

b) Describe the intersections of lines in the plane and in space, intersections of a line and a plane or of two planes in space.

c) Describe or identify conic sections and other cross sections of solids. (*Cross-sections should be of standard, familiar solids such as a sphere or cylinder, and some Platonic solids (e.g., cube, regular tetrahedron)*.)

d) Represent two-dimensional figures algebraically using coordinates and/or equations.

e) Use vectors to represent velocity and direction.

5. Mathematical Reasoning

a) Make, test, and validate geometric conjectures using a variety of methods including deductive reasoning and counterexamples.

Data Analysis and Probability

1. Data Representation

Note 1: Items should include interpretation of uncommon representation of data such as that found in newspapers and magazines.

Note 2: Bar and line graphs should increase in complexity (e.g., through more complex scale) from grade to grade.

Note 3: Representations of data for Grade 12 include histograms, line graphs, scatter plots, box plots, circle graphs, stem and leaf plots, frequency distributions, and tables. Objectives in which only a subset of these representations is applicable are indicated in the parentheses associated with the objective.

a) Read or interpret data, including interpolating or extrapolating from data.

b) For a given set of data, complete a graph and then solve a problem using the data in the graph (histograms, scatter plots, line graphs).

c) Solve problems by estimating and computing with univariate or bivariate data (including scatter plots and two-way tables).

d) Given a graph or a set of data, determine whether information is represented effectively and appropriately (bar graphs, box plots, histograms, scatter plots, line graphs).

e) Compare and contrast the effectiveness of different representations of the same data. (*For example, effects of scale change on various graphs*.)

2. Characteristics of Data Sets

a) Calculate, interpret, or use mean, median, mode, range, inter-quartile range, or standard deviation.

b Recognize how linear transformations of one-variable data affect mean, median, mode and range (e.g., effect on the mean of adding a constant to each data point).

c) Determine the effect of outliers on mean, median, mode, range, inter-quartile range, or standard deviation.

d) Compare two or more data sets using mean, median, mode, range, inter-quartile range or standard deviation describing the same characteristic for two different populations or subsets of the same population.

e) Given a set of data or a scatter plot, visually choose the line of best fit and explain the meaning of the line. Use the line to make predictions. (*Do not require calculations*.)

f) Use or interpret a normal distribution as a mathematical model appropriate for summarizing certain sets of data.

g) Given a scatter plot, make decisions or predictions involving a line or curve of best fit.

h) Given a scatter plot, estimate the correlation coefficient (e.g., Given a scatter plot, is the correlation closer to 0, .5, or 1.0? Is it a positive or negative correlation?).

3. Experiments and Samples

a) Identify possible sources of bias in data collection methods and describe how such bias can be controlled and reduced.

b) Recognize and describe a method to select a simple random sample.

c) Make inferences from sample results.

d) Identify or evaluate the characteristics of a good survey or of a well-designed experiment.

4. Probability

a) Analyze a situation that involves probability of independent or dependent events.

b) Determine the theoretical probability of simple and compound events in familiar or unfamiliar contexts.

c) Given the results of an experiment or simulation, estimate the probability of simple or compound events in familiar or unfamiliar contexts.

d) Use theoretical probability to evaluate or predict experimental outcomes.

e) Determine the number of ways an event can occur using tree diagrams, formulas for combinations and permutations, or other counting techniques.

f) Determine the probability of the possible outcomes of an event.

h) Determine the probability of independent and dependent events.

i) Determine conditional probability using two-way tables.

j) Interpret probabilities with a given context.

Algebra

1. Patterns, Relations, and Functions

a) Recognize, describe, or extend arithmetic, geometric progressions or patterns using words or symbols. (*Pattern types can include rational numbers, powers, simple recursive patterns, regular polygons, 3-dimensional shapes. Patterns can be more complex than in grade 8*.)

b) Express the function in general terms (either recursively or explicitly), given a table, verbal description, or some term of a sequence.

e) Identify or analyze distinguishing properties of linear, quadratic, inverse ($y = k/x$), or exponential functions from tables, graphs, or equations. (*Properties such as slopes, intercepts, points or intervals missing from domain*.)

g) Determine the domain and range of functions given various contexts.

h) Recognize and analyze the general forms of linear, quadratic, inverse, or exponential functions (e.g., in $y = ax + b$, recognize the roles of a and b). (*Include examining parameters and their effect on curve shape in linear and quadratic functions*.)

i) Express linear and exponential functions in recursive and explicit form given a table or verbal description

2. Algebraic Representations

a) Translate between different representations of algebraic expressions using symbols, graphs, tables, diagrams, or written descriptions. (*Can use linear, quadratic, or exponential expressions*.)

b) Analyze or interpret relationships expressed in symbols, graphs, tables, diagrams, or written descriptions. (*Includes evaluating the relative advantages or disadvantages of different representations to answer specific questions; recognizing the relationship between a linear inequality in two variables and its graph*.)

c) Graph or interpret points that are represented by one or more ordered pairs of numbers on a rectangular coordinate system. (*Includes real number coordinates, x- and y- intercepts, points of discontinuity of a graph*.)

d) Perform or interpret transformations on the graphs of linear and quadratic functions.

e) Use algebraic properties to develop a valid mathematical argument.

f) Use an algebraic model of a situation to make inferences or predictions.

g) Given a real-world situation, determine if a linear, quadratic, inverse, or exponential function fits the situation (e.g., half-life, bacterial growth).

h) Solve problems involving exponential growth and decay.

3. Variables, Expressions, and Operations

b) Write algebraic expressions, equations, or inequalities to represent a situation.

c) Perform basic operations, using appropriate tools, on algebraic expressions (including grouping and order of multiple operations involving basic operations, exponents, roots, simplifying, and expanding).

d) Write equivalent forms of algebraic expressions, equations, or inequalities to represent and explain mathematical relationships.

4. Equations and Inequalities

a) Solve linear, rational or quadratic equations or inequalities. (*No complex roots; can include real number coefficients*.)

c) Analyze situations or solve problems using linear or quadratic equations or inequalities symbolically or graphically (e.g., $ax + b = c$ and $ax + b = cx + d$.) (*No complex roots; can include real number coefficients*.)

d) Recognize the relationship between the solution of a system of linear equations and its graph.

e) Solve problems involving more advanced formulas [e.g., the volumes and surface areas of three dimensional solids; or such formulas as: $A = P(I + r)^t$, $A = Pe^{rt}$].

f) Given a familiar formula, solve for one of the variables.

g) Solve or interpret systems of equations or inequalities. (*Can be graphically or symbolically; systems of two linear or one linear and one quadratic*.)

Appendix D: 2005 NAEP Mathematics Complexity Categories

Low Complexity. This category relies heavily on the recall and recognition of previously learned concepts and principles. Items typically specify what the student is to do, which is often to carry out some procedure that can be performed mechanically. It is not left to the student to come up with an original method or solution. The following are some, but not all, of the demands that items in the low-complexity category might make:

- Recall or recognize a fact, term, or property.
- Recognize an example of a concept.
- Compute a sum, difference, product, or quotient.
- Recognize an equivalent representation.
- Perform a specified procedure.
- Evaluate an expression in an equation or formula for a given variable.
- Solve a one-step word problem.
- Draw or measure simple geometric figures.
- Retrieve information from a graph, table, or figure.

Moderate Complexity. Items in this category involve more flexibility of thinking and choice among alternatives than do those in the low-complexity category. They require a response that goes beyond the habitual, is not specified, and ordinarily has more than a single step. The student is expected to decide what to do, using informal methods of reasoning and problem solving strategies, and to bring together skill and knowledge from various domains. The following illustrate some of the demands that items of moderate complexity might make:

- Represent a situation mathematically in more than one way.
- Select and use different representations, depending on situation and purpose.
- Solve a word problem requiring multiple steps.
- Compare figures or statements.
- Provide a justification for steps in a solution process.
- Interpret a visual representation.
- Extend a pattern.
- Retrieve information from a graph, table, or figure and use it to solve a problem requiring multiple steps.
- Formulate a routine problem, given data and conditions.
- Interpret a simple argument.

High Complexity. These items make heavy demands on students, who must engage in more abstract reasoning, planning, analysis, judgment, and creative thought. A satisfactory response to the item requires that the student think in abstract and sophisticated ways. Items at the level of high complexity may ask the student to do any of the following:

- Describe how different representations can be used for different purposes.
- Perform a procedure having multiple steps and multiple decision points.
- Analyze similarities and differences between procedures and concepts.
- Generalize a pattern.
- Formulate an original problem, given a situation.
- Solve a novel problem.
- Solve a problem in more than one way.
- Explain and justify a solution to a problem.
- Describe, compare, and contrast solution methods.
- Formulate a mathematical model for a complex situation.
- Analyze the assumptions made in a mathematical model.
- Analyze or produce a deductive argument.
- Provide a mathematical justification

Appendix E: 2005 NAEP 12th Grade Mathematics Achievement Levels

Basic. *Twelfth-grade students performing at the Basic level should demonstrate procedural and conceptual knowledge in solving problems in the five NAEP content areas.* Twelfth graders performing at the Basic level should be able to use estimation to verify solutions and determine the reasonableness of results as applied to real-world problems. They are expected to use algebraic and geometric reasoning strategies to solve problems. Twelfth graders performing at the Basic level should recognize relationships presented in verbal, algebraic, tabular, and graphical forms; and demonstrate knowledge of geometric relationships and corresponding measurement skills. They should be able to apply statistical reasoning in the organization and display of data and in reading tables and graphs. They also should be able to generalize from patterns and examples in the areas of algebra, geometry, and statistics. At this level, they should use correct mathematical language and symbols to communicate mathematical relationships and reasoning processes, and use calculators appropriately to solve problems.

Proficient. *Twelfth-grade students performing at the Proficient level should consistently integrate mathematical concepts and procedures into the solutions of more complex problems in the five NAEP content areas*. Twelfth graders performing at the Proficient level should demonstrate an understanding of algebraic, statistical, and geometric and spatial reasoning. They should be able to perform algebraic operations involving polynomials, justify geometric relationships, and judge and defend the reasonableness of answers as applied to real-world situations. These students should be able to analyze and interpret data in tabular and graphical form; understand and use elements of the function concept in symbolic, graphical, and tabular form; and make conjectures, defend ideas, and give supporting examples.

Advanced. *Twelfth-grade students performing at the Advanced level should consistently demonstrate the integration of procedural and conceptual knowledge and the synthesis of ideas in the five NAEP content areas*. Twelfth-grade students performing at the Advanced level should understand the function concept and be able to compare and apply the numeric, algebraic, and graphical properties of functions. They should apply their knowledge of algebra, geometry, and statistics to solve problems in more Advanced areas of continuous and discrete mathematics. They should be able to formulate generalizations and create models through probing examples and counterexamples. They should be able to communicate their mathematical reasoning through the clear, concise, and correct use of mathematical symbolism and logical thinking.

Appendix F: Proposed Revised Mathematics Complexity Scale

Complexity of items reflects a wide variety of characteristics such as procedural vs. conceptual, focused vs. cross-cutting, concrete vs. abstract, routine vs. novel, simple vs. complicated, practical vs. theoretical, intuitive vs. formal, simple vs. complicated, mathematical vs. contextual, and so forth. To simplify analysis and reporting, NAEP consolidates all these possibilities into three fundamental categories of complexity--low, moderate, and high. Half of the NAEP assessment consists of items that can be classified as moderately complex, with the other half divided equally between items that are of low or high complexity.

Low Complexity (25%).	**Moderate Complexity (50%)**.	**High Complexity (25%)**.
Items of low complexity rely heavily on the recall of previously learned concepts and procedures. Typically, low complexity items specify what the student is to do--usually a rehearsed algorithm--and do not expect the students to devise an original method or solution. Typical expectations of items: • Recall or recognize a fact, term, or property. • Recognize an example of a concept. • Compute a sum, difference, product, or quotient. • Recognize an equivalent representation. • Perform a specified procedure. • Evaluate an expression in an equation or formula for a given variable. • Solve a one-step word problem. • Draw or measure simple geometric figures. • Retrieve information from a graph, table, or figure.	Items of moderate complexity involve flexible thinking and choice among alternatives. Moderately complex items typically require more than a single step, skill and knowledge from various domains, and responses that go beyond the habitual. Typical expectations of items: • Represent a situation mathematically in more than one way. • Select and use different representations, depending on situation and purpose. • Solve a word problem requiring multiple steps. • Compare figures or statements. • Provide a justification for steps in a solution process. • Interpret a visual representation. • Extend a pattern. • Retrieve information from a graph, table, or figure and use it to solve a problem requiring multiple steps. • Formulate a routine problem, given data and conditions. • Interpret a simple argument.	Items of high complexity require students to engage in abstract reasoning, planning, analysis, judgment, and creative thought. Such items are often stated in complicated ways that require careful thought to understand and represent mathematically. Typical expectations of items: • Describe how different representations can be used for different purposes. • Perform a procedure having multiple steps and multiple decision points. • Analyze similarities and differences between procedures and concepts. • Generalize a pattern. • Formulate an original problem, given a situation. • Solve a novel problem. • Solve a problem in more than one way. • Explain and justify a solution. • Describe, compare, and contrast solution methods. • Formulate a mathematical model for a complex situation. • Analyze the assumptions made in a mathematical model. • Analyze or produce a deductive argument. • Provide a mathematical justification

Appendix G: Suggested Changes to 2005 Grade 12 NAEP Mathematics Objectives

Left column: Objectives currently proposed in the 2005 12th grade Framework (lettered) plus proposed new objectives (in bold, with + signal). Items proposed for deletion appear in light gray as do those that lie outside the central bracket.

Right column: Proposed revisions of current objectives (in bold), explanations, examples, and references to ADP objectives.

Brackets: Objectives designated as "Low Bracket" are middle grade objectives that should be assessed at the end of grade 8 but that should also be included in the 12th grade NAEP assessment but at lower level of sampling. Objectives designated as "High Bracket" are those needed as preparation for calculus, but are not required as part of preparedness for work or credit-bearing college courses. They too should be included in the 12th grade NAEP assessment but at lower level of sampling. Both high and low bracket items appear in light gray.

Number Properties and Operations:

1.	**Number Sense**	**Comments, Examples, Proposed Revisions**
d)	Write, rename, represent, or compare real numbers (e.g., π, √2, numerical relationships using number lines, models, or diagrams).	*Clearer*: **Write, rename, represent, interpret, or compare expressions for real numbers (including absolute value, square roots, and π) using number lines, models, or diagrams. Include very large and very small numbers.** *Example*: Which is bigger: 3√2 or 2√3? (Solve by squaring, or by rewriting as √18 and √12)
f)	Represent very large or very small numbers using scientific notation in meaningful contexts.	*Clarify & combine with 1h*: **Read, write, or interpret very large or very small numbers given in scientific notation and use such numbers in meaningful contexts.** (Partly drawn from ADP I1.5.)
g)	Find or model absolute value or apply to problem situations.	*Clarify & extend:* **Interpret fractions, ratios, percentages, absolute values, square or cube roots and use in problem situations.** *Note 1:* The verb "model" is unclear since it may suggest a physical model. *Note 2:* Should be low bracket (8th grade) but usually not mastered even by grade 12; thus not classified as low bracket. *Example*: The diagonal of a unit square represents (or "models") √2 .
h)	Interpret calculator or computer displays of numbers given in scientific notation.	*Delete* as being too detailed. It is subsumed under "read" in 1f above.
j)	Order or compare real numbers, including very large or small real numbers.	*Delete*: Subsumed under 1d above.
+)	**Use the number line to locate numbers expressed in fractional or decimal form, as square or cube roots, or in scientific notation.**	*Added* to emphasize the number line; adapted from ADP I2.1 I2.2.

2.	**Estimation**	**Comments, Examples, Proposed Revisions**
a)	Establish or apply benchmarks for real numbers in contexts.	*Delete* because it is unclear. What are benchmarks?
b)	Make estimates of very large or very small numbers appropriate to a given situation by: • identifying when estimation is appropriate or not, • determining the level of accuracy needed, • selecting the appropriate method of estimation, or • analyzing the effect of an estimation method on the accuracy of results.	*Suggested simplification*: **Identify situations where estimation is appropriate, determine the required degree of accuracy, and make estimates of sufficient accuracy.** *Example*: How many sugar cubes will fill this room?
+)	**Recognize when an estimate or approximation is more appropriate than an exact answer.**	*From ADP reasoning objectives*
+)	**Analyze the relation between estimation methods and accuracy of results.**	This was the final bullet of 2.b.
c)	Verify solutions or determine the reasonableness of results in a variety of situations including scientific notation, calculator, and computer results.	*Revise*: **Verify solutions or determine the reasonableness of results for mathematical and real world problems, properly interpreting results in terms of the context.**
d)	Estimate square or cube roots of numbers less than 1,000 between two whole numbers.	*Delete*. This is a specialized detail that is subsumed under 1j above.

3. **Number Operations**	**Comments, Examples, Proposed Revisions**
a) Perform computations with real numbers including common irrational numbers or the absolute value of numbers. *(Include items that are not placed in a context and require computation with common and decimal fractions, as well as items that use a context.)*	*Suggested clarification*: **Perform computations with real numbers including rational numbers, common irrational numbers or the absolute value of numbers.** (*Rationale*: Even though rational numbers are the focus of Grade 8, they are important everywhere, and at Grade 12 the examples and problems can better reflect common uses of rational numbers)
+) Add, subtract, multiply, divide, and calculate integer powers of whole numbers, fractions and decimals without a calculator.	*Low Bracket.* Extended from ADP I1.1 to include positive exponents. Calculations without calculator should be simple. Bracketed low because it is in Grade 8 expectations. *Example*: $8^{1/3} = 2$.
d) Describe the effect of multiplying and dividing by numbers including the effect of multiplying or dividing a real number by: zero, a number less than zero, a number between zero and one, one, or a number greater than one.	
+) **Understand the effect of raising a number to a positive or negative power.**	*Example*: $(1\text{-}.05)^2 < (1\text{-}.05)$, $(1+.05)^2 > (1+.05)$, $(1\text{-}.05)^{-2} > (1\text{-}.05)$, $(1+.05)^{-2} < (1+.05)$.
g) Solve application problems involving numbers, including rational and common irrationals, using exact answers or estimates as appropriate.	*Example:* $A^2 < 1 \iff \|A\| < 1$
+) **Use calculators and computers appropriately, correctly interpret calculator and computer output, and regularly make estimations to verify technology-aided results.**	From ADP I4.1. *Note:* Objectives for effective and appropriate use of calculators need to be stressed as something more than merely allowing calculators as a tool for performing calculations.
+) **Use graphing calculators and computer spreadsheets.**	From ADP I4.2. *Note:* Use of spreadsheets is essential for work and college. Graphing calculators are of more limited value since their primary use is in mathematics instruction. *Note*: NAEP can assess "use" by asking students to predict or interpret the output of a calculator or spreadsheet display.

4. **Ratios and Proportional Reasoning**	**Comments, Examples, Proposed Revisions**
b) Use proportions to model problems.	*Delete:* Subsumed in 4c below.
c) Use proportional reasoning to solve problems (including rates).	*Better*: **Calculate, apply and reason with ratios, proportions, rates or percentages to solve problems including those involving scale drawings, similar figures, and indirect measurement.** (From ADP I1.2.)
d) Solve problems involving percentages (including percent increase and decrease, interest rates, tax, discount, tips, or part/whole relationships).	*Simplify:* **Solve problems involving percentages such as interest, tax, and tips.** *Note*: Similar ratio problems are cited below in objective 1K under Measurement.

5. **Properties of Number and Operations**	**Comments, Examples, Proposed Revisions**
b) Solve problems involving factors, multiples, or prime factorization.	*Combine* 5b and 5c: **Solve problems involving prime or composite numbers, factors, multiples, or prime factorization.**
c) Use prime or composite numbers to solve problems.	*Merge* with 5b above.
d) Use divisibility or remainders in problem settings.	*Extend* (from ADP I1.4): **Solve problems using divisibility, remainders, least common multiples, and greatest common divisors.**
e) Apply basic properties of operations. (*Properties include commutative, associative, and distributive properties of addition and multiplication*.)	*Extend and clarify*: **Apply basic properties of operations, namely the commutative, associative, distributive, and inverse properties, and the properties of identities and zero.**
+) **Correctly use conventions about the order of operations to evaluate arithmetic expressions, including those containing parentheses.**	*From ADP I1.3* Important to emphasize separately, as a convention and not a 'law.' *Example*: Why does my calculator tell me that the square of -2 is -4? (e.g. $-2^2 = -4$.)
f) Provide a mathematical argument about a numerical property or relationship.	*Clarify*: **Provide a mathematical argument using basic numerical properties to justify numerical operations and relationships.** (*Example:* Why is $\sqrt{x^2} = \|x\|$ rather than x?)
+) **Understand and work with binary notation for**	*Added objective*. Important to understand computers and

numbers.	place value notation.
+) **Understand the hierarchy of number systems — positive whole numbers, integers (positive, negative and zero), rational numbers, real numbers (rational and irrational) and complex numbers—and identify examples of each type.**	*High bracket: Added from ADP I3.*

Geometry and Measurement:

0. Measuring *Geometric* Attributes	*Note*: Formerly, this section was a major part of the Measurement content area. Since in Grade 12 geometric measurement is grouped (and scaled) with geometry, this section has been relocated to Geometry,
+) **Recognize that geometric measurements (length, area, perimeter, volume) depend on the choice of a unit, and apply such units correctly in expressions, equations and problem solutions**	*Adapted* from ADP K8.1.
c) Estimate or compare perimeters or areas of two-dimensional geometric figures.	*Extend (*based on ADP K8.2) and *combine* with 1.h below to read: **Solve real-world problems by determining, estimating or comparing perimeters or areas of two-dimensional geometric objects.** *Examples*: perimeter of a polygon, circumference of a circle; area of a rectangle, circle, triangle and a polygon.
d) Estimate or compare volumes or surface area of three-dimensional figures.	*Extend (*based on ADP K8.2), *combine* with 1.j below, and change "figure" to object: **Solve problems by determining, estimating or comparing volumes or surface areas of three-dimensional figures.** *Examples*: volume of a rectangular box, a prism, a pyramid, a cone and a sphere; surface area of a prism, a pyramid, a cone and a sphere.
e) Solve problems involving the coordinate plane such as the distance between two points, the midpoint of a segment, or slopes of perpendicular or parallel lines.	*Move* to new Geometry section on Coordinate Geometry
f) Solve problems of angle measure, including those involving triangles or other polygons or parallel lines cut by a transversal.	
h) Solve mathematical or real-world problems involving perimeter or area of plane figures such as polygons, circles, or composite figures.	*Delete*: Subsumed in 1c above.
j) Solve problems involving volume or surface area of rectangular solids, cylinders, cones, pyramids, prisms, spheres, or composite shapes.	*Delete*: Subsumed in 1d above.
k) Solve problems involving indirect measurement such as finding the height of a building by finding the distance to the base of the building and the angle of elevation to the top.	*Simplify:* Solve problems involving indirect measurement such as finding the height of a tree or a building. *Note*: These kinds of problems are really ratio problems like those of 4d above. There the examples are financial, here they are geometric.
l) Solve problems involving rates such as speed, density, population density, or flow rates.	*Move* to the new section below that deals with measurement in non-geometric contexts.
m) Use trigonometric relations in right triangles to solve problems.	*Move* to new Geometry section on Trigonometry.

1. Dimension and Shape	**Comments, Examples, Proposed Revisions**
b) Use two-dimensional representations of three-dimensional objects to visualize and solve problems involving surface area and volume. (*For example, show how doubling the radius of the bottom of a can affects the area of the label or the volume of the can*.)	
c) Give precise mathematical descriptions or definitions of geometric objects in the plane and in 3-dimensional space. (*Include full set of Platonic solids (e.g., cube, regular tetrahedron)*.)	*Note*: Changed "shapes" to "objects" because of three-dimensions.
d) Draw or sketch from a written description of plane figures (e.g., isosceles triangles, regular polygons, curved	

figures) and planar images of 3-dimensional figures (e.g., polyhedra, spheres, and hemispheres).	
e) Describe or analyze properties of spheres and hemispheres.	

2. Transformation of Shapes & Preservation of Properties	**Comments, Examples, Proposed Revisions**
a) Recognize or identify types of symmetries (e.g., point, line, rotational, self-congruences) of two- and three-dimensional figures.	
b) Give or recognize the precise mathematical relationship (e.g., congruence, similarity, orientation) between a figure and its image under a transformation.	
c) Perform or describe the effect of a single transformation on two- or three-dimensional geometric shapes (reflections across lines of symmetry, rotations, translations, dilations).	*Better:* "geometric objects," not "geometric shapes. *Add:* "…or describe the final outcome of successive transformations.
d) Describe the final outcome of successive transformations.	*Example:* The composition of two reflections across non-parallel lines gives a rotation centered at the intersection of the lines).
e) Justify relationships of congruence and similarity, and apply these relationships using scaling and proportional reasoning.	

3. Relationships between Geometric Figures	**Comments, Examples, Proposed Revisions**
b) Apply geometric properties and relationships in solving multi-step problems in two and three dimensions (including rigid and non-rigid figures).	
c) Represent problem situations with geometric models to solve mathematical or real-world problems.	
d) Use the Pythagorean theorem to solve distance problems in two- or three-dimensional situations. (*Students are expected to recall the Pythagorean theorem.*)	*Add* "and its converse" (based on ADP K5): "**Use the Pythagorean theorem and its converse to solve problems involving indirect measurement and distance in …**"
e) Describe and analyze properties of circles (e.g., perpendicularity of tangent and radius, angle inscribed in a semicircle).	*Reorganization of (e), (f), (g)* (from ADP K2, K3, and K4): **Understand definitions and basic properties of, and solve problems about, congruent and similar triangles, circles, quadrilaterals, polygons, perpendicular and intersecting lines, and associated angle relationships.**
+) **Identify and apply properties of and theorems about angles and parallel and perpendicular lines.**	*Adapted* from ADP K2.1, K2.2, K2.3.
f) Analyze properties or relationships of triangles, quadrilaterals, and other polygonal plane figures.	*High bracket (remaining parts of reorganization):* **Know and prove basic theorems about congruent and similar triangles, circles, quadrilaterals, polygons, perpendicular and intersecting lines, and associated angle relationships.**
g) Describe or analyze properties and relationships of parallel, perpendicular or intersecting lines including the angle relationships that arise in these cases.	Delete: Included above.

4. Position and Direction	**Comments, Examples, Proposed Revisions**
b) Describe the intersections of lines in the plane and in space, intersections of a line and a plane or of two planes in space.	
c) Describe or identify conic sections and other cross sections of solids. (*Cross-sections should be of standard, familiar solids such as a sphere or cylinder, and some Platonic solids (e.g., cube, regular tetrahedron).*)	
e) Use vectors to represent velocity and direction.	*Extend*: **Use vectors to represent velocity and direction; compute multiples and sums of vectors, both algebraically and graphically.**

5. Trigonometry:	*Note:* A new section largely derived from ADP.
+) **Understand the definitions of sine, cosine and tangent**	Adapted from ADP K11.1

as ratios of sides in a right triangle and use these functions to solve problems about length of sides, measure of angles, and area of general triangles.	
() Understand how similarity of right triangles allows the trigonometric functions sine, cosine and tangent to be properly defined as ratios of sides.	High bracket. From ADP K11.1. This is about the mathematical notion of being "well-defined," an esoteric concept that only mathematicians worry about.
() Apply the trigonometric functions sine, cosine and tangent to solve for an unknown length of a side of a right triangle, given one of the acute angles and the length of another side.	From ADP K11.2, but redundant since it is subsumed by "solve problems" in the first objective above.
() Use the standard formula for the area of a triangle, A = 1/2bh, to explain the area formula, A = 1/2absinC where a and b are the lengths of two sides of a triangle and C is the measure of the included angle formed by these two sides, and use it to find the area of a triangle when given the lengths of two of its sides and the included angle.	From ADP K11.3, but redundant since it is subsumed by "solve problems" in the first objective above.
+) **Recognize that periodic behavior can be represented by a sine wave and understand how the amplitude, frequency, and phase shift indicate characteristics of that behavior.**	*Note*: Intended as a limited objective appropriate for the goal of readiness for "all students." It is not sufficiently mathematical for the high bracket (pre-calculus) goal, but nonetheless may be beyond the main bracket.
+) Understand radian measure for angles as the arc length of the circumference of a circle of radius 1 subtended by a central angle.	High bracket.
+) Use radian measure to extend the graphs of trigonometric functions to periodic functions on the real line and use these functions to solve problems.	High bracket. Adapted from asterisked ADP K12.
+) **Understand, interpret, and use the identity $\sin^2\theta + \cos^2\theta = 1$ for angles θ between 0˚ and 90˚.**	*Extracted* from asterisked ADP K12.1. *Note*: This is an interpretation of the Pythagorean theorem.
+) Know and use other trigonometric identities such as addition and double angle formulas.	High bracket. Adapted from asterisked ADP K12.2.
+) Graph sine, cosine, and tangent as well as their reciprocals, secant, cosecant, and cotangent; identify key characteristics.	High bracket. From asterisked ADP K12.3.
+) Know and use the law of cosines and the law of sines to find missing sides and angles of a triangle.	High bracket. From asterisked ADP K12.4/

6. Coordinate Geometry	*Note:* Also a new section.
+) **Use the coordinate plane to solve problems involving the distance between two points, the midpoint of a line segment, or slopes of perpendicular or parallel lines.**	*Moved* and adapted from NAEP Measurement section; similar to ADP K10.3
d) Represent two-dimensional figures algebraically using coordinates and/or equations.	Extend (from ADP K10): **Represent two-dimensional geometric figures algebraically using coordinates; use algebra to solve geometric problems.**
+) **Use the coordinates of two points on a line to define its slope, and use slope to express the parallelism and perpendicularity of lines.**	From ADP K10.1. *Note*: Since this objective is in Grades 7-8 of the MAP expectations, it should logically be placed in a Low bracket for the 12th Grade NAEP. However, since it is still not common in those grades, it is left here in the central bracket.
+) Describe a line by a linear equation.	*Ignore*. From ADP K10.2. *Note*: This is too vague to be useful, and is subsumed in several objectives under Algebra.
+) Find an equation of a circle given its center and radius and, given an equation of a circle, find its center and radius.	High bracket. From ADP K10.4.
+) Graph ellipses and hyperbolas whose axes are parallel to the x and y axes and demonstrate understanding of the relationship between their standard algebraic form and their graphical characteristics.	High bracket. From ADP J4.6.

7. Mathematical Reasoning	*Note:* **A new section.** Even though mathematical reasoning cuts across all of mathematics, the major objectives are

	located in Geometry because this is the subject where the notion of theorem and proof have historically been introduced.
+) **Understand and correctly use relational conjunctions (“and,” “or,” “not”), terms of causation (“if… then”) and equivalence (“if and only if”).**	
+) **Understand converse and contrapositive; recognize examples of flawed reasoning such as "Since A ⇒ B, therefore B ⇒ A”**	
+) Understand syllogisms, tautologies, circular reasoning.	*High bracket.*
+) **Using inductive and deductive reasoning to arrive at valid conclusions.**	
+) **Understanding the role of definitions, proofs and counter-examples in mathematical reasoning; constructing simple proofs.**	
a) Make, test, and validate geometric conjectures using a variety of methods.	*Add inductive reasoning*: " ... **including inductive reasoning, deductive reasoning, and counterexamples."**
+) **Understand the role of axioms, definitions, theorems, and counter-examples in the logical structure of mathematics; identify and give examples of each.**	From ADP K1.1.
+) Understand and explain a geometric proof of the Pythagorean Theorem.	*High bracket.* Added to strengthen the presence of proofs as a step towards the goal of logical reasoning.
+) **Recognize that there are geometries, other than Euclidean geometry, in which the parallel postulate is not true.**	From ADP K1.3
+) State and prove basic theorems in geometry.	*High bracket.* From ADP K1.2, ADP K2.1, K2.2, and K2.3. *Note:* List of specific theorems has been omitted.
+) Perform straight edge and compass constructions such as a line parallel to a given line through a point not on the line; the perpendicular bisector of a line segment; and the bisector of an angle.	*High bracket.* From ADP K2.1, K2.2, and K2.3.

Data Analysis and Probability:

0. Non-Geometric Measurement	*Note:* Some objectives relocated from Measurement
+) Understand that numerical values associated with measurements of physical quantities are approximate and must be assigned units of measurement.	From ADP K8. *Note*: Should be low bracket, but students do not know this.
+) Solve problems involving rates such as speed, density, population density, or flow rates.	*Example*: To make a vegetable soup in a 6-quart cylindrical pot, you start by boiling 1 gallon of water. In order to avoid spilling as it boils, you must not fill the pot over 90% of its capacity. Assuming that the density of the vegetables you use is 2.44 lb/qt, what is the maximum weight of vegetables you may use?
a) Select or use appropriate type of unit for the attribute being measured such as volume or surface area.	*Delete* "such as" clause so as not to restrict this objective to geometric measurement. It should apply to time, money, temperature, etc., as well as to length, area, and volume.
b) Solve problems involving conversions within or between common measurement systems, given the relationship between the units. (*Can include cubic units, and rates such as miles per hour to feet per second*.)	
e) Determine appropriate accuracy of measurement in problem situations (e.g., the accuracy of measurement of the dimensions to obtain a specified accuracy of area) and find the measure to that degree of accuracy. (*For example, how accurately must we measure a piece of paper that is between 7 and 8 inches wide and 11 and 12 inches long to determine its area to 1/10 of a square inch accuracy?)*	*Simplify:* **Determine accuracy of measurement in problem situations required to obtain a specified accuracy of result.** *Note*: Similar to the new 2b under Number which includes all kinds of estimation (not just measurement).

f) Construct or solve problems (e.g., number of rolls needed for insulating a house) involving scale drawings.	Delete: Part of Number 4c.
g) Compare lengths, areas or volumes of similar figures using proportions and scale factors.	Revised from ADP K8.3.

1. Data Representation	**Comments, Examples, Proposed Revisions**
Note 1: Items should include interpretation of uncommon representation of data such as that found in newspapers and magazines.	*Note:* **Data generally refer to numbers with units. To the extent practical, items should employ authentic data such as that found in the media.**
Note 3: Representations of data for grade 12 include histograms, line graphs, scatter plots, box plots, circle graphs, stem and leaf plots, frequency distributions, and tables. Objectives in which only a subset of these representations is applicable are indicated in the parentheses associated with the objective.	*Note:* **Data may be represented by histograms, line graphs, scatter plots, box plots, circle graphs, stem and leaf plots, frequency distributions, x-bar charts and tables.**
a) Read or interpret data, including interpolating or extrapolating from data.	*Extend* (ADP L1.2): **Read or interpret data, tables, charts, and graphs, including interpolating or extrapolating from data.**
+) Organize and display data using appropriate methods (including spreadsheets) to detect patterns and departures from patterns.	From ADP L1.1
b) For a given set of data, complete a graph and then solve a problem using the data in the graph (histograms, scatter plots, line graphs).	*Revise*: **Given a set of data and an associated but incomplete graph, complete the graph and then solve a problem using the data in the graph.**
c) Solve problems by estimating and computing with univariate or bivariate data (including scatter plots and two-way tables).	
+) **Create scatter plots, analyze patterns and describe relationships in paired data**.	From ADP L1.5.
d) Given a graph or a set of data, determine whether information is represented effectively and appropriately (bar graphs, box plots, histograms, scatter plots, line graphs).	
e) Compare and contrast the effectiveness of different representations of the same data. (For example, effects of scale change on various graphs.)	

2. Characteristics of Data Sets	**Comments, Examples, Proposed Revisions**
a) Calculate, interpret, or use mean, median, mode, range, inter-quartile range, or standard deviation.	*Revise and expand:* **Calculate, interpret, or use summary statistics for distributions of data including measures of center (mean, median) and spread (range, inter-quartile range, percentiles, variance, standard deviation).** From ADP L1.3
b) Recognize how linear transformations of one-variable data affect mean, median, mode and range (e.g., effect on the mean of adding a constant to each data point).	*Revise:* **Recognize how linear transformations of one-variable data affect mean, median, mode, range, and inner quartile range**. *Example:* What is the effect on the mean of adding a constant to each data point.
c) Determine the effect of outliers on mean, median, mode, range, inter-quartile range, or standard deviation.	
d) Compare two or more data sets using mean, median, mode, range, inter-quartile range or standard deviation describing the same characteristic for two different populations or subsets of the same population.	*Simpler* alternative (from ADPL1.4): **Compare data sets using graphs and summary statistics.**
e) Given a set of data or a scatter plot, visually choose the line of best fit and explain the meaning of the line. Use the line to make predictions. (*Do not require calculations*.)	*Better:* **Construct a scatter plot of a set of paired data and find the line of best fit (regression line) either by visual estimation or by using a graphing calculator.** (Adapted from ADP L3.4)
f) Use or interpret a normal distribution as a mathematical model appropriate for summarizing certain sets of data.	*Expanded*: **Understand the Gaussian normal distribution as a mathematical model appropriate for summarizing**

	certain sets of data; know and interpret the key characteristics of this distribution. From ADP L1.6.
g) Given a scatter plot, make decisions or predictions involving a line or curve of best fit.	*Better*: **Explain when it is (and when it is not) appropriate to use the regression line to make predictions and use the regression line to make appropriate predictions.** (Adapted from ADP L3.4)
h) Given a scatter plot, estimate the correlation coefficient (e.g., Given a scatter plot, is the correlation closer to 0, .5, or 1.0? Is it a positive or negative correlation?).	*Expanded*: **Understand that the correlation coefficient of bivariate data is a number between -1 and +1 that measures the direction and goodness of fit of the regression line; visually estimate the correlation coefficient (e.g., positive or negative, closer to 0, .5, or 1.0) of a scatter plot.** (Adapted from ADP L3.4.)

3. Experiments and Samples	**Comments, Examples, Proposed Revisions**
a) Identify possible sources of bias in data collection methods and describe how such bias can be controlled and reduced.	*Expand* (based on ADP L3.1): **Identify possible sources of bias in data collection methods, including the impact of sampling methods and the phrasing of questions asked; describe how such bias can be reduced and controlled; and explain the impact of such bias on the conclusions that can rightfully be made.**
b) Recognize and describe a method to select a simple random sample.	
+) **Explain the differences in design and in warranted conclusions between randomized experiments and observational studies.**	From ADP L3.3.
c) Make inferences from sample results.	
d) Identify or evaluate the characteristics of a good survey or of a well-designed experiment.	
+) **Design simple experiments or investigations to collect data to answer questions of interest.**	From ADP L3.2.

4. Probability	**Comments, Examples, Proposed Revisions**
+) **Understand how probability quantifies the likelihood that an event occurs and why all probabilities must be between 0 and 1.**	From ADP L4.1
a) Analyze a situation that involves probability of independent or dependent events.	
b) Determine the theoretical probability of simple and compound events in familiar or unfamiliar contexts.	
c) Given the results of an experiment or simulation, estimate the probability of simple or compound events in familiar or unfamiliar contexts.	
+) **Explain how the relative frequency of a specified outcome of an event can be used to estimate the probability of the outcome.**	From ADP L4.2
d) Use theoretical probability to evaluate or predict experimental outcomes.	
e) Determine the number of ways an event can occur using tree diagrams, formulas for combinations and permutations, or other counting techniques.	
f) Determine the probabilities of the possible outcomes of an event.	
h) Determine the probabilities of independent and dependent events.	*Example*: Stress counter-intuitive examples such as finding the probability that among 5 randomly chosen individuals 2 share the same birthday (day and month).
+) **Understand the law of large numbers and apply it to simple examples.**	From ADP L4.3.
i) Determine conditional probability using two-way tables.	
j) Interpret probabilities with a given context.	*Alternative*: **Interpret and apply probability concepts (including conditional probability and independent events) to practical situations to make informed decisions.**

	(From ADP L4.4 and L4.5.)
+) Understand the binomial theorem and its connections to combinatorics, Pascal's triangle, and probability.	High bracket. From ADP J6.

5. Use and Interpret Evidence	*Note*: Objectives related to real-world problem solving
+) **Identify misleading uses of data and critique different ways of presenting and using information.**	*Adapted from* ADP L2 and L 2.2.
+) **Evaluate published reports by considering the source of the data, the design of the study, and the way the data are analyzed and displayed.**	*Adapted from* ADP L2.1.
+) **Distinguish relevant from irrelevant information, identify missing information, and either find what is needed or make appropriate estimates.**	*From ADP reasoning section*
+) **Recognize and use the process of mathematical modeling when mathematical structures are embedded in other contexts.**	*From ADP reasoning section*
+) **Recognize when arguments based on data confuse correlation with causation.**	ADP L2.3.

Algebra:

0. Sets and Boolean Algebra	*Note:* A new section, listed first because it is new, not because it belongs first.
+) **Understand and use the concepts of sets, elements, empty set, relations (e.g. belong to), subsets. Recognize different ways to define sets (lists, algebraic property, inductive).**	
+) **Understand and use operations on sets: union, intersection, complement; apply to on-line search techniques.**	
+) **Understand finite and infinite sets, the concept of cardinality**.	*High bracket.*
+) **Interpret Venn diagrams and use them to solve problems**.	

1. Patterns, Relations, and Functions	**Comments, Examples, Proposed Revisions**
+) **Understand the difference between relation and function, and determine whether a relationship given in verbal, symbolic or graphical form is a function**.	*Expanded* from ADP J2.1. *Example:* F(son)=father *is* a function but S(father)=son is not.
a) Recognize, describe, or extend arithmetic, geometric progressions or patterns using words or symbols. (*Pattern types can include rational numbers, powers, simple recursive patterns, regular polygons, 3-dimensional shapes.*)	*Caution*: Pattern problems are quite controversial among some mathematicians who argue that questions about extending patterns without a clearly stated underlying rule are non-mathematical. However, in workplace settings, as well as in many college courses, this kind of inductive reasoning is of primary importance.
+) **Understand function notation and evaluate a function at a specified point in its domain.**	ADP J2.3.
b) Express the function in general terms (either recursively or explicitly), given a table, verbal description, or some terms of a sequence.	
e) Identify or analyze distinguishing properties of linear, quadratic, reciprocal ($y = k/x$), or exponential functions from tables, graphs, or equations. (*Properties such as slopes, intercepts, points or intervals missing from domain*.)	*Add to parentheses*: **"Properties such as general shapes, slopes, ..."** *Note*: It is better to call $f(x) = k/x$ a *reciprocal* function to avoid confusion with the more general idea of f^{-1} as an *inverse* function. (Better still, might focus on *power* functions $f(x) = x^p$ where p = -2, -1, 0, ½, 1, 2, 3.)
g) Determine the domain and range of functions given various contexts.	*Revised:* **Understand the concepts of domain and range, and determine the domain of functions given in various forms and contexts.** (Part of ADP J2.2.)

+) Determine the range of functions given in various forms and contexts.	High bracket (since range is sometimes difficult to calculate). The second part of ADP J2.2.
h) Recognize and analyze the general forms of linear, quadratic, reciprocal, or exponential functions (e.g., in $y = ax + b$, recognize the roles of a and b). (*Include examining parameters and their effect on curve shape in linear and quadratic functions*.)	*Note*: Changed "inverse" to "reciprocal" here.
+) **Recognize, interpret, and graph piecewise and step functions, including those that employ the absolute value function $f(x) = \|x\|$.**	
i) Express linear and exponential functions in recursive and explicit form given a table or verbal description.	
+) Using function notation, combine functions by addition, subtraction, multiplication and division.	High bracket (because of function notation). Part of ADP J2.4.
+) Combine functions by composition,	High bracket. Part of ADP J2.4. Strangely, this is more often taught than the preceding.
+) Identify whether a function has an inverse and when functions are inverses of each other, explain why the graph of a function and its inverse are reflections of one another over the line y = x.	High bracket. From J2.5. *Examples*: squaring and square root; exponential and log. *Note*: Here "inverse" is used correctly.
+) Know that the inverse of an exponential function is a logarithm, prove basic properties of a logarithm using properties of its inverse and apply those properties to solve problems.	High bracket. From ADP J2.6 Example: If f(t) = the population in year t, what does f^{-1} (3000) = 1965 mean?

2. Algebraic Representations	Comments, Examples, Proposed Revisions
+) **Using multiple representations to represent problems and solutions.**	*From ADP Reasoning section.*
+) **Use the special symbols of mathematics correctly and precisely.**	*From ADP Reasoning section.*
a) Translate between different representations of algebraic expressions using symbols, graphs, tables, diagrams, or written descriptions. (*Can use linear, quadratic, or exponential expressions*.)	*Note:* A similar new objective appears in section 3 below.
b) Analyze or interpret relationships expressed in symbols, graphs, tables, diagrams, or written descriptions. (*Includes evaluating the relative advantages or disadvantages of different representations to answer specific questions; recognizing the relationship between a linear inequality in two variables and its graph*.)	
c) Graph or interpret points that are represented by one or more ordered pairs of numbers on a rectangular coordinate system. (*Includes real number coordinates, x- and y- intercepts, points of discontinuity of a graph*.)	*Delete*: The first part is subsumed under coordinate Geometry, and the discontinuity part is confusing. In any case, it is not of fundamental importance.
d) Perform or interpret transformations on the graphs of linear and quadratic functions.	
+) **Understand and explain why some transformations commute and some do not.**	*Example:* Reflecting across the x-axis and then rotating 90° is not the same as rotating 90° and then reflecting across the x-axis.
e) Use algebraic properties to develop a valid mathematical argument.	
f) Use an algebraic model of a situation to make inferences or predictions.	
g) Given a real-world situation, determine if a linear, quadratic, inverse, or exponential function fits the situation (e.g., half-life, bacterial growth).	*Extend*: **Given a real-world situation, determine if a linear, quadratic, reciprocal, inverse, exponential or logarithmic function fits the situation.** *Examples*: projectile motion, half-life, bacterial growth, earthquakes.
h) Solve problems involving exponential growth and decay.	

3. Variables, Expressions, and Operations	Comments, Examples, Proposed Revisions
	Note: To avoid perpetuating confusion of "expression" with "equation," we suggest moving all objectives concerning

	equations and inequalities from this section to the following Section 4 whose title is "Equations and Inequalities."
b) Write algebraic expressions, equations, or inequalities to represent a situation.	*Better*: **Read and write algebraic expressions and interpret their form in relation to diverse contexts.**
c) Perform basic operations, using appropriate tools, on algebraic expressions (including grouping and order of multiple operations involving basic operations, exponents, roots, simplifying, and expanding).	*Simplify*: **Know when and how to transform algebraic expressions and do so fluently and accurately by correctly employing basic operations such as grouping, simplifying, and expanding**. Partly adapted from ADP J1.
d) Write equivalent forms of algebraic expressions, equations, or inequalities to represent and explain mathematical relationships.	*Better:* **Understand when one representation of an algebraic expression may be more appropriate than another for a particular purpose; choose different but equivalent forms of algebraic expressions to observe different properties of the same mathematical relationship.**
+) **Understand the properties of exponents and roots and apply these properties to interpret and simplify algebraic expressions.**	Adapted from ADP J1.1and ADP J1.2
+) **Combine, manipulate, simplify, and evaluate polynomials and simple rational expressions.**	From ADP J1.3, J1.4, J1.5, and J1.6.
+) **Express algebraic relations symbolically, verbally, numerically, and graphically and translate among these different representations.**	*Note*: This gets at understanding, and is much more important than the preceding item which is only about calculation.
+) Recognize and use formulas for terms of arithmetic and geometric series.	Adapted from ADP J1.7
+) Recognize, derive, and use the formulas for the sum of finite arithmetic and geometric series and for the sum of an infinite geometric series whose common ratio r is in the interval (-1, 1).	High bracket. Adapted from ADP J1.7

4. Equations and Inequalities	**Comments, Examples, Proposed Revisions**
+) **Read & write equations & inequalities and interpret their forms in relation to diverse given contexts.**	Relocated from Section 3 above (to place equations in Section 4).
+) **Choose different but equivalent forms of equations and inequalities to observe different properties of the same mathematical relationship.**	Relocated from Section 3 above (to place equations in Section 4).
a) Solve linear, rational or quadratic equations or inequalities. (*No complex roots; can include real number coefficients*.)	*Revise*: **Solve linear, rational or quadratic equations or inequalities with real roots, including those involving the absolute value of a linear function.** (ADP J3, J3.1)
+) **Solve systems of two linear equations in two variables.**	*Adapted* from ADP J3.3.
d) Recognize the relationship between the solution of a system of linear equations and its graph.	*Better*: **Understand the relationship between the solution of a system of two linear equations in two variables and the graphs of the corresponding lines.** From ADP J4.3
c) Analyze situations or solve problems using linear or quadratic equations or inequalities symbolically or graphically (e.g., $ax + b = c$ and $ax + b = cx + d$.) (*No complex roots; can include real number coefficients*.)	Revise**: Understand the relationship between the equation $f(x) = g(x)$ and the intersection points of the graphs of the functions f and g.** *Examples:* Intersections of a line and a parabola, or a line and a circle.
+) Solve quadratic equations in one variable, including complex roots.	High bracket (complex roots). From J3.5.
+) Solve systems of three linear equations in three variables by row reduction.	High bracket. Adapted from ADP J3.4
+) Recognize, interpret, and solve problems that can be modeled using *linear equations* in one variable, (time/rate/distance problems); *systems of two equations* in two variables (mixture problems); *quadratic equations*, falling objects); *exponential functions* and *logarithms* (compound growth and decay); and *finite geometric series* (home mortgage).	From ADP J5.
e) Solve problems involving special formulas [e.g., the volumes and surface areas of common three dimensional solids, or such formulas as: $A = P(I + r)^t$, $A = Pe^{rt}$].	Note: This requires cube roots, fractional exponents (e.g. cube roots for volumes) and calculators.
f) Given a familiar formula, solve for one of the variables.	*Better* (since it avoids the subjective term "familiar"): **Solve**

	an equation involving several variables for one variable in terms of the others. (ADP J3.2)
+) **Demonstrate strategic competence in using equations and expressions.**	*Examples:* (A) Suppose you need to find the maximum value of $5t - 3t^2 - 2$. Which [of the following] is the best first step to take in order to achieve this goal? (B) Dividing by 2 is the wrong first step to employ if your goal is to solve 2x + 3 = 4x + 9.
+) **Predict the form of a solution from the form of an equation.**	*New.*
+) **When solving problems, think about strategy, test ideas, try different approaches, check for errors and reasonableness of solutions, and devise ways to verify results.**	*From ADP reasoning section. Really about problem solving.*
+) **Shift regularly between the specific and the general, use examples to understand general ideas, and extend specific results to more general cases to gain insight.**	*From ADP reasoning section. Really about problem solving.*
g) Solve or interpret systems of equations or inequalities. (*Can be graphically or symbolically; systems of two linear or one linear and one quadratic*.)	*Delete*: Covered by Algebra 2.b and 4.d and other items above.

5. Graphs and Functions	**Comments, Examples, Proposed Revisions**
+) **Graph a linear function and demonstrate that it has a constant rate of change.**	From ADP J4.1; also related to ADP K10.2
+) **Understand the relationship between the coefficients of a linear equation and the slope and x- and y-intercepts of its graph.**	From ADP J4.2 *Note*: Duplicative of earlier items.
+) **Graph the solution set of a system of one, two, or three linear inequalities.**	*Simplified* from ADP J4.4.
+) **Graph a quadratic function and understand the relationship between its real zeros and the x-intercepts of its graph.**	From ADP J4.5.
+) **Graph exponential and logarithm functions (base 2 or 10) and identify their key characteristics**.	*Extended* from ADP J4.7. *Note*: Applications of logarithm function include pH, earthquakes (Richter scale), and decibels.
+) **Read information from graphs, identify properties that provide useful information, and draw appropriate conclusions.**	*Adapted* from ADP J4.8.

Appendix H: Proposed Revised Grade 12 NAEP Mathematics Objectives

(-) *Low Bracket*. Middle grade objectives that should be re-assessed in the 12th grade NAEP but at lower level of sampling.

(*) *High Bracket*. Objectives necessary to prepare for calculus but are not part of preparedness for work or credit-bearing college courses. They should be included in the 12th grade NAEP assessment but at lower level of sampling.

Note: Frequently in these objectives the conjunction "or" is used where "and" seems more appropriate (e.g., "read, write, or interpret" in Number 1b below). This usage is the residue of earlier NAEP objectives where each item is seen not as an objective for student learning (where "and" is usually most appropriate) but as an instruction to item writers who are being told to test any one of the named options.

Number Properties and Operations

1. *Number Sense*

a. Write, rename, represent, interpret, or compare expressions for real numbers using number lines, models, or diagrams (including absolute value, square and cube roots, π, fractional or decimal forms, scientific notation, very large and very small numbers). *Example*: Which is bigger: $3\sqrt{2}$ or $2\sqrt{3}$? (Solve by squaring, or by rewriting as $\sqrt{18}$ and $\sqrt{12}$).

b. (-) *Read, write, or interpret very large or very small numbers given in scientific notation and use such numbers in meaningful contexts.*

c. Interpret fractions, ratios, percentages, absolute values, square or cube roots, and common logarithms, and use in problem situations. *Examples*: $\sqrt{2}$ can be represented as the diagonal of a unit square. Because $10^2 = 100$, $\log_{10}100 = 2$.

2. *Estimation*

a. Identify situations where estimation is appropriate, determine the required degree of accuracy, and make estimates of sufficient accuracy.

b. Recognize when an estimate or approximation is more appropriate than an exact answer.

c. Analyze how estimation methods influence the accuracy of results.

d. Verify solutions or determine the reasonableness of results for mathematical and real world problems, properly interpreting results in terms of the context.

e. Understand the concept of order of magnitude and how it is related to place value.

f. Understand how the decimal system can express arbitrarily close approximations to physical quantities.

3. *Number Operations*

a. *(-) Add, subtract, multiply, divide, and calculate simple integral and fractional powers of whole numbers, fractions and decimals, using a calculator when appropriate*. *Example*: $8^{1/3} = 2$.

b. Perform computations with real numbers including rational numbers, common irrational numbers or the absolute value of numbers.

c. Describe the effect of multiplying and dividing including the effect of multiplying or dividing a real number by: zero, a number less than zero, a number between zero and one, one, or a number greater than one. *Examples:* $A^2 < 1 \iff |A| < 1$; $-3x < 1 \iff x > -1/3$

d. Analyze the effect of raising a number to a positive or negative power. *Examples*: $(1-.05)^2 < (1-.05)$, $(1+.05)^2 > (1+.05)$, $(1-.05)^{-2} > (1-.05)$, $(1+.05)^{-2} < (1+.05)$.

e. Solve application problems involving numbers, including rational and common irrationals, using exact answers or estimates as appropriate.

f. Use graphing calculators and computer spreadsheets, interpret calculator and computer output, and make estimations to verify technology-aided results.

4. Ratios and Proportional Reasoning

a. (-) *Solve problems involving percentages such as interest, tax, and tips*.

b. Calculate, apply and reason with ratios, proportions, rates or percentages to solve problems including those involving scale drawings, similar figures, and indirect measurement.

5. Properties of Number and Operations

a. (-) *Solve problems involving prime or composite numbers, factors, multiples, or prime factorization*.

b. Solve problems using divisibility, remainders, least common multiples, and greatest common divisors.

c. Apply basic properties of operations, namely the commutative, associative, distributive, and inverse properties, and the properties of the additive and multiplicative identities 1 and 0.

d. (-) *Correctly use conventions about the order of operations to evaluate arithmetic expressions, including those containing parentheses*. *Example*: Why does my calculator tell me that the square of -2 is -4 (e.g. $-2^2 = -4$)?

e. Provide a mathematical argument using basic numerical properties to justify numerical operations and relationships. *Example:* Why is $\sqrt{x^2} = |x|$ rather than x?

f. Understand and work with binary notation for numbers.

g. (*) *Understand the hierarchy of number systems—positive whole numbers, integers (positive, negative and zero), rational numbers, real numbers (rational and irrational) and complex numbers—and identify examples of each type.*

h. (*) *Know the difference between rational and irrational numbers and understand a proof that $\sqrt{2}$ is irrational.*

Geometry and Measurement

1. Measuring Geometric Attributes

a. Recognize that geometric measurements (length, area, perimeter, volume) depend on the choice of a unit, and apply such units correctly in expressions, equations and problem solutions.

b. (-) *Understand and apply the effect of proportions and scaling on length, areas and volume.* Example: Show how doubling the radius of the bottom of a can affects the area of the label or the volume of the can.

c. Solve real-world problems by determining, estimating or comparing perimeters or areas of two-dimensional geometric objects. *Examples*: perimeter of a polygon, circumference of a circle; area of a rectangle, circle, triangle and a polygon.

d. Solve problems by determining, estimating or comparing volumes or surface areas of three-dimensional figures. *Examples*: volume of a rectangular box, a prism, a pyramid, a cone and a sphere; surface area of a prism, a pyramid, a cone and a sphere.

e. Solve problems of angle measure, including those involving triangles or other polygons or parallel lines cut by a transversal.

f. Solve problems involving indirect measurement such as finding the height of a tree or a building.

2. Dimension and Shape

a. (-) *Use two-dimensional representations of three-dimensional objects to visualize and solve problems involving surface area and volume*.

b. Give precise mathematical descriptions or definitions of geometric objects in the plane and in 3-dimensional space, including all Platonic solids.
c. Draw or sketch from a written description plane figures (e.g., isosceles triangles, regular polygons, curved figures) and planar images of 3-dimensional figures (e.g., polyhedra, spheres, and hemispheres).
d. Translate between polyhedra and their associated nets.
e. Describe or analyze properties of spheres and hemispheres.

3. Transformation of Shapes & Preservation of Properties

a. (-) *Recognize or identify types of symmetries (point, line, rotational) of two- and three-dimensional figures.*
b. Give or recognize the precise mathematical relationship (e.g., congruence, similarity, orientation) between a figure and its image under a transformation.
c. Perform or describe the effect of transformations on two- or three-dimensional geometric objects (reflections across lines of symmetry, rotations, translations, dilations).
d. (*) *Describe the final outcome of successive transformations*. *Example:* Explain why the composition of two reflections across non-parallel lines gives a rotation centered at the intersection of the lines.
e. Justify relationships of congruence and similarity, and apply these relationships using scaling and proportional reasoning.

4. Relationships between Geometric Figures

a. Apply geometric properties and relationships in solving multi-step problems in two and three dimensions (including rigid and non-rigid figures).
b. Represent problem situations with geometric models to solve mathematical or real-world problems.
c. Use the Pythagorean theorem and its converse to solve problems involving indirect measurement and distance in two- or three-dimensional situations.
d. Understand definitions and basic properties of, and solve problems about, congruent and similar triangles, circles, quadrilaterals, polygons, parallel, perpendicular and intersecting lines, and associated angle relationships.
e. (*) *Know and prove basic theorems about congruent and similar triangles, circles, quadrilaterals, polygons, perpendicular and parallel lines, and associated angle relationships.*
f. (*) *Analyze and explain one of the geometric proofs of the Pythagorean Theorem*.
g. Recognize that there are geometries unlike Euclidean geometry in which the parallel postulate is not true.
h. (*) *Understand the hypotheses, conclusions, and various proofs of basic theorems in geometry*.
i. (*) *Perform straight edge and compass constructions such as the line parallel to a given line through a point not on the line; the perpendicular bisector of a line segment; and the bisector of an angle.*

5. Position and Direction

a. Describe the intersections of lines in the plane and in space, intersections of a line and a plane or of two planes in space.
b. Describe or identify conic sections and other cross sections of familiar solids.
c. (*) *Use vectors to represent velocity and direction; compute multiples and sums of vectors, both algebraically and graphically*.

6. Trigonometry

a. Understand the definitions of sine, cosine and tangent as ratios of sides in a right triangle and use them to solve problems about length of sides, measure of angles, and area of general triangles.
b. Understand how similarity of right triangles allows the trigonometric functions sine, cosine and tangent to be properly defined as ratios of sides.
c. Use graphs to display and interpret periodic behavior and understand how amplitude, frequency, and phase shifts represent characteristics of that behavior.
d. Understand, interpret, and use the identity $\sin^2\theta + \cos^2\theta = 1$ for angles θ between 0° and 90°; recognize this identity as a special representation of the Pythagorean theorem.
e. (*) *Understand radian measure for an angle as the arc length of the circumference of a circle of radius 1 subtended by that central angle.*
f. (*) *Use radian measure to extend the graphs of trigonometric functions to periodic functions on the real line and use these functions to solve problems.*
g. (*) *Know and use other trigonometric identities such as addition and double angle formulas.*
h. (*) *Graph sine, cosine, and tangent as well as their reciprocals, cosecant, secant, and cotangent; identify key characteristics.*
i. (*) *Know and use the law of cosines and the law of sines to find unknown sides and angles of a triangle.*

7. Coordinate Geometry

a. Solve problems in the coordinate plane involving the distance between two points, the midpoint of a line segment, or slopes of perpendicular or parallel lines.
b. Represent two-dimensional geometric figures algebraically using coordinates; use algebra to solve geometric problems.
c. Find an equation of a circle given its center and radius and, given an equation of a circle, find its center and radius.
d. (*) *Graph ellipses and hyperbolas whose axes are parallel to the coordinate axes and demonstrate understanding of the relationship between their standard algebraic form and their graphical characteristics.*
e. (*) *Calculate the centroid of a collection of points via their coordinates.*

Data Analysis and Probability

1. Measurement and Data

a. (-) *Understand that numerical values associated with measurements of physical quantities are approximate, subject to variation, and must be assigned units of measurement.*
b. (-) *Select or use appropriate type of unit for the attribute or variable being measured*.
c. Solve problems involving rates such as speed, density, population density, or flow rates.
d. Solve problems involving conversions within or between common measurement systems, given the relationship between the units. *Example*: Miles per hour to feet per second.
e. Determine accuracy of measurement in problem situations required to obtain a specified accuracy of result. *Example*: How accurately must we measure a piece of paper that is between 7 and 8 inches wide and 11 and 12 inches long to determine its area to 1/10 of a square inch accuracy?

2. Data Representation

a. Understand that data generally refer to numbers with units.
b. Recognize the difference between real ("authentic") data and artificial ("invented") data.

c. Understand the difference between measurement data (length, temperature) and categorical data (gender, marital status).
d. Identify and critique authentic data such as those found in the media.
e. Interpret graphical representations of data including histograms, line graphs, scatter plots, box plots, circle graphs, stem and leaf plots, frequency distributions, x-bar charts, and tables.
f. Interpret numerical and symbolic representations of data including tables, charts, and diagrams, including interpolating or extrapolating from data.
g. Recognize and explain the dangers of extrapolating from data.
h. Organize and display data using appropriate methods (including graphs, tables, and spreadsheets) to detect patterns and departures from patterns.
i. Given a set of data and an associated but incomplete graph, complete the graph and then solve a problem using the data in the graph.
j. Solve problems involving univariate categorical and measurement data.
k. Solve problems involving bivariate categorical data using two-way tables.
l. Create and use scatter plots to analyze patterns and describe relationships involving bivariate measurement data.
m. Given a graphical or tabular representation of a set of data, determine whether information is represented effectively and appropriately.
n. Compare and contrast different graphical representations of univariate data. *Example*: Effects of origin location and scale change on various graphs.
o. Explain the effect of scale changes on bivariate data when displayed in scatter plots..

3. ***Characteristics of Data Sets***

a. Calculate, interpret, or use summary statistics for distributions of data including measures of typical value (mean, median), position (quartiles, percentiles), and spread (range, inter-quartile range, variance, standard deviation).
b. Recognize how linear transformations of one-variable data affect mean, median, mode, range, and inner quartile range. *Example*: What is the effect on the mean of adding a constant to each data point?
c. Determine the effect of outliers on mean, median, mode, range, inter-quartile range, or standard deviation. *Example:* Mean income for some populations may be very different from the median income, usually much higher. (For this reason median rather than mean is often used when reporting income.)
d. Compare data sets using graphs and summary statistics.
e. Construct a scatter plot of a set of bivariate data, approximate a trend line if a linear pattern is apparent, and when it is appropriate for the data use a graphing calculator to construct a least squares regression line (line of best fit) .
f. Understand the Gaussian (normal) distribution as a mathematical model appropriate for summarizing (approximating) certain sets of data; know and interpret the key characteristics of this distribution such as shape, center (mean), spread (standard deviation) and location (z-scores).
g. Explain when it is (and when it is not) appropriate to use the regression line to make predictions and use the regression line to make appropriate predictions.
h. Understand that the correlation coefficient of bivariate data is a number between –1 and +1 that measures the direction and goodness of fit of the regression line; visually estimate the correlation coefficient (e.g., positive or negative, closer to 0, .5, or 1.0) of a scatter plot.

4. *Experiments and Samples*

a. Identify possible sources of bias in sample surveys including those due to sampling methods and those due to measurement errors in collecting data (such as biased phrasing of questions).
b. Describe how bias in sample surveys can be reduced and controlled and explain the impact of such bias on the conclusions that can rightfully be made.
c. Recognize and describe a method to select a simple random sample.
d. Identify possible sources of bias (or confounding) in experiments and describe how such bias can be controlled.
e. Explain the differences in design and in warranted conclusions between randomized experiments and observational studies.
f. Draw warranted inferences from samples (e.g., estimates of proportions in a whole population, estimates of population means, decision about differences in means for two "treatments").
g. Identify and evaluate the characteristics of a good survey or of a well-designed experiment.
h. Design simple experiments or surveys to collect data to answer questions of interest.

5. *Probability*

a. Understand how probability quantifies the likelihood that an event occurs and why all probabilities must be between 0 and 1.
b. Decide whether or not two events are independent of each other.
c. Determine the theoretical probability of simple and compound events in familiar or unfamiliar contexts.
d. Given the results of an experiment or simulation, estimate the probability of simple or compound events in familiar or unfamiliar contexts.
e. Explain how the relative frequency of occurrences of a specified outcome of an event can be used to estimate the probability of the outcome.
f. Determine the number of ways an event can occur using tree diagrams, formulas for combinations and permutations, or other counting techniques.
g. Determine the probabilities of events. *Example*: Emphasize counter-intuitive examples such as the probability that among several randomly chosen individuals two share the same birthday.
h. Understand the law of large numbers and apply it to simple examples of estimating unknown probabilities.
i. Determine conditional probability using two-way tables.
j. Interpret and apply probability concepts (including conditional probability and independent events) to practical situations to make informed decisions.
k. (*) *Understand the binomial theorem and its connections to combinatorics, Pascal's triangle, and probability*.

6. *Use and Interpret Evidence*

a. Identify misleading uses of data in real-world settings and critique different ways of presenting and using information.
b. Evaluate published reports by considering the source of the data, the design of the study, and the way the data are analyzed and displayed.
c. Distinguish relevant from irrelevant information, identify missing information, and either find what is needed or make appropriate approximations.
d. Recognize, use, and distinguish the processes of mathematical (deterministic) and statistical modeling.
e. Understand the difference between an observational study and a randomized, controlled experiment.

f. Recognize when arguments based on data confuse correlation with causation.

Algebra

1. *Mathematical Reasoning*

a. Understand and correctly use relational conjunctions ("and," "or," "not"), terms of causation ("if… then") and equivalence ("if and only if").

b. Understand converse and contrapositive; recognize examples of flawed reasoning such as "Since $A \Rightarrow B$, therefore $B \Rightarrow A$"

c. (*) *Understand syllogisms, tautologies, circular reasoning.*

d. Understand the difference between inductive and deductive reasoning, namely that induction creates conjectures whereas deduction can prove (or refute) conjectures.

e. Make, test, and validate conjectures using a variety of methods including inductive reasoning, deductive reasoning, and counterexamples.

f. Understand the role of axioms, definitions, theorems, and counter-examples in the logical structure of mathematics; identify and give examples of each.

2. *Sets and Boolean Algebra*

a. Understand and use the concepts of sets, elements, empty set, relations (e.g. belong to), subsets. Recognize different ways to define sets (lists, defining property, inductive).

b. Understand and use operations on sets: union, intersection, complement. *Example:* Use Boolean search techniques to refine on-line bibliographic searches.

c. Interpret Venn diagrams and use them to solve problems.

d. (*) *Understand finite and infinite sets, the concept of cardinality*.

3. *Patterns, Relations, and Functions*

a. Understand the difference between relation and function, and determine whether a relationship given in verbal, symbolic or graphical form is a function. *Example: F*(son)=father *is* a function but *S*(father)=son is not.

b. Recognize, describe, or extend arithmetic, geometric progressions or patterns using words or symbols. (Pattern types can include rational numbers, powers, simple recursive patterns, regular polygons, 3-dimensional shapes.)

c. Understand function notation and evaluate a function at a specified point in its domain.

d. Understand the concepts of domain and range, and determine the domain of functions given in various forms and contexts.

e. Use graphs or algebraic reasoning to informally determine the range of simple functions in various forms and contexts. *Example:* The range of $f(x) = 3(x-1)^2 + 5$ is $[5,\infty)$.

f. Recognize and analyze the general forms of linear, quadratic, reciprocal, or exponential functions (e.g., in $y = ax + b$, recognize the roles of a and b). (Include examining parameters and their effect on curve shape in linear and quadratic functions.)

g. Recognize, interpret, and graph piecewise and step functions, including those that employ the absolute value function $f(x) = |x|$.

h. Express linear and exponential functions in recursive and explicit form given a table, verbal description, or some terms of a sequence.

i. (*) *Using function notation, combine functions by addition, subtraction, multiplication and division.*

j. (*) *Combine functions by composition.*

k. (*) *Understand the concept of an inverse function, identify whether a function has an inverse, and determine when functions are inverses of each other*. *Example:* If $f(t)$ = the population in year t, what does $f^{-1}(3000) = 1965$ mean?

l. (*) *Explain why the graph of a function and its inverse are reflections of one another over the line $y = x$*. *Examples*: Squaring and square root; exponential and log.

m. (*) *Know that the inverse of an exponential function is a logarithm, prove basic properties of a logarithm using properties of its inverse and apply those properties to solve problems*.

4. *Algebraic Representations*

a. Use the special symbols of mathematics correctly and precisely. *Examples:* $\sum$ $\leq$ $\cup$ $\in$ $\perp$

b. Translate between different representations of linear, quadratic, or exponential expressions using symbols, graphs, tables, diagrams, or written descriptions.

c. Analyze or interpret relationships expressed in symbols, graphs, tables, diagrams, or written descriptions; evaluate the relative advantages or disadvantages of different representations to answer specific questions.

d. Know the definition of the graph of a linear inequality in two variables and be able to graph such inequalities.

e. Perform or interpret transformations on the graphs of linear, quadratic, and exponential functions.

f. Understand and explain why some transformations commute and some do not. *Example*: Reflecting across the x-axis and then rotating 90° is not the same as rotating 90° and then reflecting across the x-axis.

g. Use algebraic properties to develop a valid mathematical argument. *Example:* Justify the steps involved in solving an equation by appeal to properties of operations.

h. Make inferences or predictions using an algebraic model of a situation.

i. Given a real-world situation, determine if a linear, quadratic, reciprocal, inverse, exponential or logarithmic function fits the situation. *Examples*: Projectile motion, half-life, bacterial growth, Richter scale for earthquakes, logarithmic scales in graphs.

j. Solve problems involving exponential growth and decay.

k. Understand (and illustrate via spreadsheets) that constant first differences characterize linear behavior, constant second differences characterize quadratic behavior, constant ratios characterize exponential behavior.

5. *Variables, Expressions, and Operations*

a. Read and write algebraic expressions and interpret their form in relation to diverse contexts.

b. Know when and how to transform algebraic expressions and do so fluently and accurately by correctly employing basic operations such as grouping, factoring, combining, and expanding.

c. Understand when one representation of an algebraic expression may be more appropriate than another for a particular purpose; choose different but equivalent forms of algebraic expressions to observe different properties of the same mathematical relationship.

d. Understand the properties of exponents and roots (e.g., the law of exponents) and apply these properties to interpret and simplify algebraic expressions.

e. Evaluate algebraic expressions, including polynomials and rational functions.

f. Express algebraic relations symbolically, verbally, numerically, and graphically and translate among these different representations.

g. Understand the relationship between terms in arithmetic and in geometric series and be able to generate these terms when these relationships are specified numerically.

h. (*) Recognize, derive, and use the formulas for the sum of finite arithmetic and geometric series.

i. (*) *Recognize, derive, and use the formulas for the sum of an infinite geometric series whose common ratio r is in the interval (-1, 1).*

6. *Equations and Inequalities*

a. Read and write equations and inequalities and interpret the form in which they are expressed in relation to diverse given contexts.
b. Choose different but equivalent forms of equations and inequalities to observe different properties of the same mathematical relationship.
c. Solve linear, rational or quadratic equations or inequalities with real roots, including those involving the absolute value of a linear function.
d. Solve systems of two linear equations in two variables.
e. Understand the relationship between the solution of a system of two linear equations in two variables and the graphs of the corresponding lines.
f. Understand the relationship between the equation $f(x) = g(x)$ and the intersection points of the graphs of the functions f and g. *Examples:* Intersections of a line and a parabola, or a line and a circle.
g. (*) *Solve quadratic equations with complex roots*.
h. (*) *Solve systems of three linear equations in three variables and know how to do so by row reduction*.
i. Recognize, interpret, and solve problems that can be modeled using *linear equations* in one variable (time/rate/distance problems); *systems of two equations* in two variables (mixture problems); *quadratic equations* (falling objects); *exponential functions* and *logarithms* (compound growth and decay); and *finite geometric series* (home mortgage).
j. Solve problems involving special formulas [e.g., the volumes and surface areas of common three dimensional solids, or such formulas as: $A = P(I + r)^t$, $A = Pe^{rt}$].
k. Solve an equation involving several variables for one variable in terms of the others.
l. Demonstrate competence in selecting and employing strategies to transform expressions and solve equations.
m. Predict the existence or characteristics of a solution from the form of an equation. Examples: If a and b are positive, the solution to $ax + b = 0$ must be negative. Similarly, $3 + (x+1)^2 = 2 - (x-5)^2$ has no solutions because the left side is always ≥ 3 while the right side is always ≤ 2.
n. When solving problems, think about strategy, test ideas, try different approaches, check for errors and reasonableness of solutions, and devise ways to verify results.
o. Shift regularly between the specific and the general, use examples to understand general ideas, and extend specific results to more general cases to gain insight.

7. *Graphs and Functions*

a. Graph a linear function and demonstrate that it has a constant rate of change.
b. Understand the relationship between the coefficients of a linear equation and the slope and x- and y-intercepts of its graph.
c. Graph the solution set of a system of one, two, or three linear inequalities.
d. Graph a quadratic function and understand the relationship between its real zeros, the x-intercepts of its graph, and concavity.
e. (*) *Graph exponential and logarithm functions (base 2, 10 and e) and identify their key characteristics.*
f. Read information from graphs, identify properties that provide useful information, and draw appropriate conclusions.

g. Identify and analyze distinguishing graphical properties of linear, quadratic, reciprocal ($y = k/x$), or exponential functions from tables, graphs, or equations. *Examples*: Polynomials, rational functions, slopes, intercepts, increasing or decreasing, end-point behavior, points or intervals missing from domain.